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ON GEOMETRY OF CR- WARPED PRODUCT SUBMANIFOLDS OF A SEMI SYMMETRIC NON METRIC CONNECTION IN A NEARLY LORENTZIAN PARA- SASAKIAN MANIFOLD

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ABSTRACT In this paper, generalised CR- Warped product Submanifolds of a Nearly Lorentzian Para-Sasakian Manifolds with semi symmetric non metric connection are studied. Moreover, general shorp inequality of the CR-Warped product Submanifolds on Nearly Lorentzian Para-Sasakian Manifolds with semi symmetric non metric connection have been discussed.

KEYWORDS: Contact CR-submanifold, Warped product, nearly Lorentzian para-Sasakian manifold, and semi symmetric non metric connection.

1. INTRODUCTION

The concept of warped product manifolds was given by Bishop and O'Neill [5].These manifolds generalize the Riemannian product manifolds and appear in differential geometric studies in natural way [7, 10]. Later on, CR-warped product in Kaehler manifolds were studied by Chen, B.Y and showed many interesting results on the existence of warped product and proved general sharp inequalities for the second fundamental form in terms of the warping function f[8]. Arslan, K. and et al. [4], Bonanzinga, V., Matsumoto, K. [6] and Mihai, I. [13] studied for the same inequalities in almost Hermitian as well as almost contact metric manifolds. K. Lorentzian para-Sasakian manifold were studied by Matsumoto, K [12]. Then Mihai, I et al. [14] introduced the same notion independently and they obtained several results on this manifolds. Lorentzian para-Sasakian manifolds have also been studied by Matsumoto, K et al. [7], De, U.C et al., [9], and others. Lorentzian para-Sasakian manifolds with different connection was studied by several authors in ([1, 2, 15–17]). In [19], the author and et al., studied semi-invariant submanifolds of a nearly Lorentzian para-Sasakian manifold. Contact CR-warped product submanifolds of nearly Lorentzian para-Sasakian manifold were studied by Rahman [17].

In the present paper we extend the conservativeness of warped product contact CR-submanifolds of nearly Lorentzian para-Sasakian manifolds with semi symmetric non metric connection. After preliminaries in section 2, we study in section 3 and prove some existence and non existence results and then obtain a general sharp inequality for the second fundamental form in terms of the warping function f and Sasakian and cosymplectic on nearly Lorentzian para-Sasakian manifolds with semi symmetric non metric connection. In section 4 we obtained the inequality is more general as it generalizes all inequalities obtained for contact CR-warped products in contact metric manifolds.

2. PRELIMINARIES

Let $\bar{M}$ be an n dimensional almost contact metric manifold with almost contact metric structure (ϕ,ξ,η,g) such that

$$\phi^2X = X+\eta(X)\xi,\ \eta(\xi)=-1,\ g(X,\xi)=\eta(X) \tag{1}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y),$$
$$g(\phi X,Y)=g(X,\phi Y)=\psi(X,Y) \tag{2}$$

For vector fields X,Y tangent to M.Then the structure(ϕ,ξ,η,g) zienpara-contact structure.Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi=0,\quad \eta(\phi X)=0\ \text{ and } rank(\phi)=n-1$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian) manifold if

$$(\bar{\nabla}_X\phi)Y = g(X,Y)\xi+\eta(Y)X+2\eta(X)\eta(Y)\xi \tag{3}$$

$$\bar{\nabla}_X\xi=\phi X \tag{4}$$

For all vector fields X,Y tangent to $\bar{M}$ where $\bar{\nabla}$ is the Riemannian connection with respect to g. Further, an almost contact metric manifold $\bar{M}$ on (ϕ,ξ,η,g) is called nearly Lorentzian para-Sasakian if

$$(\bar{\nabla}_X\phi)Y+(\bar{\nabla}_Y\phi)X = 2g(X,Y)\xi+4\eta(X)\eta(Y)$$
$$+\eta(X)Y+\eta(Y)X \tag{5}$$

The covariant derivative of the tensor field $\phi$is defined as

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y-\phi\bar{\nabla}_XY \tag{6}$$

On the other hand semi symmetric non metric connection$\bar{\nabla}$ defined by

$$\bar{\nabla}_XY=\nabla_XY+\eta(Y)X \tag{7}$$

Substituting (1), (2) and (4) in (6) and (7) respectively, we get

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y-\phi\bar{\nabla}_XY$$

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$$= \nabla_X \phi Y + \eta(\phi Y)X - \phi \nabla_X Y - \eta(Y)\phi X$$
$$= (\nabla_X \phi)Y + \phi(\nabla_X Y) - \phi \nabla_X Y - \eta(Y)\phi X$$
$$= (\nabla_X \phi)Y - \eta(Y)\phi X$$
$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi - \eta(Y)\phi X \quad (8)$$

In particular, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly Lorentzian Para- Sasakian manifold $\bar{M}$ with semi symmetric non metric connection if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 2g(X,Y)\xi + 4\eta(X)\eta(Y)\xi + \eta(Y)X + \eta(X)Y - \eta(Y)\phi X - \eta(X)\phi Y \quad (9)$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (10)$$
$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N + \eta(N)X \quad (11)$$

For any $X, Y \in TM$ and $N \in T^{\perp}M$ where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X,Y), N)$$

Let (N_1, g_1) and (N_2, g_2) be two Riemannian Manifolds and f be the positive differentiable function on N_1. The warped product of $N_1 and N_2$ is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where [5] $g = g_1 + f^2 g_2$. A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function f is constant.

We recall the following general result for later use.

Lemma 2.1 ([5]). Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function, then

(i) $\nabla_X Y \in \Gamma(TN_1)$ is the lift of $\nabla_X Y$ on N_1,

(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$

(iii) $\nabla_Z \omega = \nabla_Z^{N_2} \omega - g(Z, \omega)\nabla \ln f$

for each $X, Y \in \Gamma(TN_1)$ and $Z, \omega \in \Gamma(TN_2)$, where $\nabla \ln f$ is the gradient of lnf and ∇ and ∇^{N_2} denote the Levi Civita connections on M and N_2, respectively.

For a Riemannian manifold M of dimension n and a smooth function f on M, we recall ∇f , the gradient of f which is defined by

$$g(\nabla f, X) = X(f) \quad (12)$$

for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{n} (e_i(f))^2 \quad (13)$$

for an orthonormal frame $\{e_1, \ldots, e_n\}$ on M.

3. CONTACT CR-WARPED PRODUCT SUBMANIFOLDS

In this section first we recall the invariant, anti-invariant and contact CR-submanifolds.For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds.We mention the following:

(i) A submanifold M tangent to ξ is an invariant submanifold if ϕ preserves any tangent space of M, that is $\phi(T_p M) \subset T_p M$,for every $p \in M$.

(ii) A submanifold M tangent to ξ is an anti-invariant submanifold if ϕ maps any tangent space of M into the normal space, that is, $\phi(T_p M) \subset T_p^{\perp} M$. for every $p \in M$.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$, then for every $p \in M$ there exists a maximal invariant subspace denoted by D_p of the tangent space $T_p M$ of M if the dimension of D_p is same for all values of $p \in M$, then D_p gives an invariant distribution D on M.

A submanifold M of an almost contact manifold $\bar{M}$ is said to be a contact CR submanifold if there exists on M a differentiable distribution D whose orthogonal complementary distribution $D^{\perp}$ is antivariant, that is;

(i) $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$

(ii) $D$is invariant distribution,i.e. $\phi D \subseteq TM$,

(iii) $D^{\perp}$ is an anti-invariant distribution i.e. $\phi D^{\perp} \subseteq T^{\perp}M$.

A contact CR-submanifold is an anti-invariant if $D_p = \{0\}$ and invariant if $D_p^{\perp} = \{0\}$respectively,for every $p \in M$.It is a proper contact CR-submanifold if neither $D_p = \{0\}$ nor $D_p^{\perp} = \{0\}$, for every$p \in M$.

If v is the ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then in case of contact CR- submanifold , the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \phi D^{\perp} \oplus v$$

where$v$ and ϕ-invariant normal sub bundle of $T^{\perp}M$.

We investigate in this section the warped products $M = N_{\perp} \times_f N_T$ and $M = N_T \times_f N_{\perp}$ where N_T and $N_{\perp}$ are invariant and anti-invariant Submanifolds of a nearly Lorentzian para-Sasakian manifold $\bar{M}$,respectively.First we discuss the warped products $M = N_{\perp} \times_f N_T$, here two possible cases arises:

(i) ξ is tangent to N_T,

(ii) ξ is tangent to $N_{\perp}$.

Now we begin with the case (i)

Theorem3.1. Let $\bar{M}$ be a nearly Lorentzian para-Sasakian manifold with a semi symmetric non metric connection $\bar{M}$. Then there do not exist warped product submanifold $M = N_{\perp} \times_f N_T$ such that N_T is an invariant submanifold tangent to ξ and $N_{\perp}$ is an anti invariant submanifold, unless $\bar{M}$ is nearly Sasakian.

Proof: Consider $\xi \in \Gamma(TN_T)$and $Z \in \Gamma(TN_{\perp})$, then by the structure equation of nearly Lorentzian para-Sasakian manifold, $(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 2g(X,Y)\xi + 4\eta(X)\eta(Y)\xi + \eta(Y)X + \eta(X)Y - \eta(Y)\phi X - \eta(X)\phi Y$, we have

$$(\bar{\nabla}_Z \phi)\xi + (\bar{\nabla}_\xi \phi)Z = 2g(Z,\xi)\xi + 4\eta(Z)\eta(\xi)\xi + \eta(\xi)Z + \eta(Z)\xi - \eta(\xi)\phi Z - \eta(Z)\phi Z$$

Using (6), we obtain

$$\bar{\nabla}_\xi \phi Z - 2\phi h(Z,\xi) = -Z + \phi Z \quad (14)$$

Taking the inner product with ϕZ in (14) and then using (2) and the fact that $\xi \in \Gamma(TN_T)$, we get $\|Z\|^2 = 0$ and hence we conclude that M is invariant, which proves the theorem.

Now, we will discuss the other case, when ξ is tangent to $N_{\perp}$.

Theorem 3.2. Let $\bar{M}$ be a nearly Lorentzian para-Sasakian manifold with a semi symmetric non metric connection. Then there do not exist warped product submanifolds $M = N_{\perp} \times_f N_T$ such that $N_{\perp}$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\bar{M}$, unless $\bar{M}$ is nearly Kenmotsu.

Proof: Let $\xi \in \Gamma(TN_T)$and$X \in \Gamma(TN_\perp)$, $(\bar{\nabla}_X\phi)\xi + (\bar{\nabla}_\xi\phi)X = -X + \phi X$ Using (6), we get

$$-\phi(\bar{\nabla}_X\xi) + \bar{\nabla}_\xi\phi X - \phi\bar{\nabla}_\xi X = -X + \phi X \quad (15)$$

Taking the inner product with X in (15) and using (2), (11), Lemma 2.1 (ii) and the fact thatξ is tangent to $N_\perp$,we obtain $\|X\|^2 = 0$. Thus, we conclude that M is anti-invariant submanifold of a nearly Lorentzian para-Sasakian manifold $\bar{M}$ otherwise $\bar{M}$ is nearly sasakian. This completes the proof.

Now, we will discuss the warped product $M = N_T \times_f N_\perp$ such that the structure vector field ξ is tangent to

Theorem 3.3. Let $\bar{M}$ be a nearly Lorentzian para-Sasakian manifold with a semi symmetric non metric connection. Then there do not exist warped product submanifolds $M = N_T \times_f N_\perp$such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\bar{M}$.

Proof: If we consider $\xi \in \Gamma(TN_T)$and $X \in \Gamma(TN_\perp)$, then we have $(\bar{\nabla}_X\phi)\xi + (\bar{\nabla}_\xi\phi)X = -X + \phi X$. Using (6), we obtain

$$-\phi\bar{\nabla}_X\xi + \bar{\nabla}_\xi\phi X - \phi\bar{\nabla}_\xi X = -X + \phi X$$

Then by (2.5) and Lemma 2.1 (ii), we derive

$$(\phi X\, lnf)\xi - 2\phi h(X,\xi) + h(\phi X,\xi) = -X + \phi X \quad (16)$$

Hence, the result is obtained by taking the inner product with ξ in (16). If we consider the structure vector field ξ tangent to N_T for the warped product $M = N_T \times_f N_\perp$, then we prove the following result for later use.

Lemma: 3.1. Let $M = N_T \times_f N_\perp$be a contact CR-warped product submanifold of a nearly Lorentzian para-Sasakian with a quarter symmetric non metric connection M such that N_T and $N_\perp$are invariant and antiinvariant submanifolds of $\bar{M}$, respectively. Then, we have

(i) $\xi\,(\ln f) = 1$,

(ii) $g(h(X,Y),\phi Z) = 0$.

(iii) $g(h(X,\omega),\phi Z) = g(h(X,Z),\phi\omega)$
$= -\frac{1}{3}\{\eta(X)g(Z,\omega) - (\phi X lnf)g(Z,\omega)\}$

(iv) $3g(h(\xi,Z),\phi\omega) = g(Z,\omega)$.

for every $X, Y \epsilon\, \Gamma(TN_T)$ and $Z,\omega\, \epsilon\, \Gamma(TN_\perp)$

Proof: If ξ is tangent to N_T,for any $Z\epsilon\Gamma(TN_\perp)$, we have $(\bar{\nabla}_\xi\phi)Z + (\bar{\nabla}_Z\phi) = -Z + \phi Z$. Then from (2.6), (2.7) and Lemma 2.1(ii), we have

$$\bar{\nabla}_\xi\phi Z + \phi\left(\nabla_\xi Z + h(Z,\xi)\right) + \phi\left(\nabla_Z\xi + h(Z,\xi)\right) = -Z + \phi Z$$

$$2(\xi lnf)\phi Z + 2\phi h(Z,\xi) - \bar{\nabla}_\xi\phi Z = -Z + \phi Z. \quad (17)$$

Taking the inner product with ϕZ in (17) and using (2) we drive

$$2(\xi lnf)\|Z\|^2 - g\left(\bar{\nabla}_\xi\phi Z,\phi Z\right) = \|Z\|^2 \quad (18)$$

On the other hand, by the property of Riemannian connection, we have $\xi g(\phi Z,\phi Z) = 2g\left(\bar{\nabla}_\xi\phi Z,\phi Z\right)$. By (2) and the property of Riemannian connection we get,

$$g(\bar{\nabla}_\xi Z,Z) = g\left(\bar{\nabla}_\xi\phi Z,\phi Z\right) \quad (19)$$

Using this fact in (18) and then from (7) $[\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)]$ and Lemma 2.1 (ii) $[\nabla_X Z = \nabla_Z X = (X\ln f)Z]$, we deduce that $(\xi\ln f)\|Z\|^2 = \|Z\|^2 \Rightarrow \xi(lnf) = 1$ for any $Z\,\epsilon\,\Gamma(TN_\perp)$, which gives (i) of lemma (3.1). For the other part of the Lemma, we have

$$(\bar{\nabla}_X\phi)Z + (\bar{\nabla}_z\phi)X = 2g(X,Z)\xi + \eta(Z)X + \eta(X)Z + 4\eta(X)\eta(Z)\xi - \eta(Z)\phi X - \eta(X)\phi Z.$$

for any $X\,\epsilon\,\Gamma(TN_T)$ and $Z\,\epsilon\,\Gamma(TN_\perp)$. Using to (6), (11) & (12), we drive

$$\begin{aligned}\bar{\nabla}_X\phi\, Z - \phi(\bar{\nabla}_X Z) + \bar{\nabla}_z\phi X - \phi\bar{\nabla}_z X &= -A_{\phi z}X + \nabla_X^\perp\phi Z + \eta(\phi Z)X - \phi\nabla_X Z\\ &-\phi h(X,Z) + \nabla_Z\phi X + h(\phi X,Z) - \phi\nabla_Z X\\ &-\phi h(X,Z) = -A_{\phi z}X + \nabla_X^\perp\phi Z - 2(Xlnf)\phi Z\\ &+(\phi Xlnf)Z + h(\phi X,Z) - 2\phi h(X,Z)\\ &= \eta(X)Z - \eta(X)\phi Z \quad (20)\end{aligned}$$

Thus, the second part can be obtained by taking the inner product in (3.8) with Y, for any $Y\epsilon\,\Gamma(TN_T)$. Again taking the inner product in (3.8) with ω for any $\omega\,\epsilon\,\Gamma(TN_\perp)$, we get

$$\eta(X)g(Z,Y) - \eta(X)g(\phi Z,Y) = -g\left(A_{\phi z}X,Y\right) + g(\nabla_X^\perp\phi Z,Y) - 2(Xlnf)g(\phi Z,Y) + (\phi Xlnf)g(Z,Y) + g(h(\phi X,Z),Y) - 2\phi g(h(X,Z),Y)$$

$$0 = -g\left(A_{\phi z}X,Y\right) \Rightarrow g(h(X,Y),\phi Z) = 0$$

lemma (3.1) (ii)

$$\eta(X)g(Z,\omega) - \eta(X)g(\phi Z,\omega) = -g\left(A_{\phi z}X,\omega\right) + g(\nabla_X^\perp\phi Z,\omega) - 2(Xlnf)g(\phi Z,\omega) + \phi(X\ln f)g(Z,\omega) + g(h((\phi X,Z),\omega) - 2\phi g(h(X,Z),\omega)$$

$$\eta(X)g(Z,\omega) = -g(h(X,\omega),\phi Z) + (\phi Xlnf)g(Z,\omega) - 2g(h(X,Z),\phi\omega) \quad (3.9)$$

By polarization identity, we get

$$\eta(X)g(Z,\omega) = -g(h(X,Z),\phi\omega) + (\phi X\ln f)g(Z,\omega) - 2g(h(X,\omega),\phi Z) \quad (3.10)$$

Then from (3.9) and (3.10), we have ,

$$g(h(X,\omega),\phi Z) + (\phi X\ln f)g(Z,\omega) - 2g(h(X,Z),\phi\omega) = -g(h(X,Z),\phi\omega) + (\phi X\ln f)g(Z,\omega) - 2g(h(X,\omega),\phi Z)$$

$$2g(h(X,\omega),\ \phi Z) - g(h(X,\omega),\ \phi Z) = -g(h(X,Z),\phi\omega) + 2g(h(X,Z),\phi\omega)$$

$$g(h(X,\omega),\ \phi Z) = g(h(X,Z),\phi\omega) \quad (3.11)$$

Which is the first equality of (iii). Using (3.11) in (3.9) or in (3.10),

Putting $g(h(X,\omega),\phi Z) = g(h(X,Z),\phi\omega)$ in (3.10) we get the second equality of (iii) $\eta(X)g(Z,\omega) = -g(h(X,\omega),\phi Z) + (\phi X\ln f)g(Z,\omega) - 2g(h(X,\omega),\phi Z)$

$$= (\phi X\ln f)g(Z,\omega) = -3g(h(X,\omega),\phi Z)$$

$$3g(h(X,\omega),\phi Z) = (\phi X\ln f)g(Z,\omega) - \eta(X)g(Z,\omega)$$

$$g(h(X,\omega),\ \phi Z) = \tfrac{1}{3}\{(\phi X\ln f)g(Z,\omega) - \eta(X)g(Z,\omega)\}$$

$$g(h(X,\omega)\,,\phi Z) = g(h(X,Z),\phi\omega) = -\tfrac{1}{3}\{\eta(X)g(Z,\omega) - (\phi X\ln f)g(Z,\omega)\}$$

Which is lemma 3.1 (iii) .

Thus

$$3g(h(X,Z)\,,\phi\omega) - (\phi\xi\ln f)g(Z,\omega) = -\eta(X)g(Z,\omega)$$

Put $X = \xi$, we get

$$3g(h(\xi,Z),\phi\omega) - (\phi\xi\ln f)g(Z,\omega) = -\eta(\xi)g(Z,\xi)$$

$$3g(h(\xi,Z),\phi\omega) = g(Z,\omega)$$

Which is lemma 3.1 (iv) thus we have proved all part of lemma 3.1

Now, we have the following characterization theorem.

Theorem 3.4. Let M be a contact CR-submanifold of a nearly Lorentz para-Sasakian manifoldwith a semi symmetric non metric connection $\bar{M}$ with integrable invariant and anti-invariant distribution $D\oplus\langle\xi\rangle$ and $D^\perp$. Then M is locally a contact CR-warped product if any only if the shape operator of M satisfies

$$A_{\phi\omega}X = \tfrac{1}{3}\{(\phi X\mu)\omega - \eta(X)\omega\},\ \forall\ X\,\epsilon\Gamma(D\oplus\langle\xi\rangle),\ \omega\,\epsilon\Gamma(D^\perp)$$

For some smooth function μ on M satisfying $V(\mu) = 0\ for\ every\ V\ \epsilon\Gamma(D^{\perp})$.

Proof: Direct part of the follows from the Lemma 3.1 (iii)

$$g(h(X,\omega),\phi Z) = g(h(X,Z),\phi\omega)$$
$$= -\frac{1}{3}\{\eta(X)g(Z,\omega) - (\phi X lnf)g(Z,\omega)\}\eta(X)g(Z,\omega)$$
$$-(\phi X lnf)g(Z,\omega)$$
$$= -g(h(X,Z),\phi\omega) - 2\,g(h(X,\omega),\phi Z).$$

Putting

$$g(h(X,Z),\phi\omega) = -g(h(X,Z),\phi\omega) - 2\,g(h(X,\omega),\phi Z)$$
$$\eta(X)g(Z,\omega) - (\phi X lnf)g(Z,\omega) =$$

Put $\mu = lnf$

$$A_{\phi\omega}X = \frac{1}{3}\{(\phi\mu X)\omega - \eta(\omega)X\} \quad (23)$$
$$g(h^*(Z,\omega),\phi X) = g\big((\overline{\nabla}_Z\phi)\omega, X\big) + g(A_{\phi\omega}X, \quad (24)$$
$$= g\big((\overline{\nabla}_Z\phi)\omega, X\big) + \frac{1}{3}\{\phi X\mu) - \eta(X)\}g(Z,\omega) \quad (25)$$
$$2g(h^*(Z,\omega),\phi X) = g\big((\overline{\nabla}_Z\phi)\omega + (\overline{\nabla}_\omega\phi)Z, X\big)$$
$$+\frac{2}{3}\{(\phi X)\mu - \eta(X)\}g(Z,\omega) \quad (26)$$
$$(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = 2g(X,Y)\xi + 4\eta(X)\eta(Y)\xi$$
$$+\eta(Y)X + \eta(X)Y - \eta(Y)\phi X - \eta(X)\phi Y$$

Replace $X = Z,\quad Y = \omega$

$$(\overline{\nabla}_Z\phi)\omega + (\overline{\nabla}_\omega\phi)Z = 2g(Z,\omega)\xi + \eta(\omega)Z$$
$$+\eta(Z)\omega + 4\eta(Z)\eta(\omega) - \eta(\omega)\phi Z - \eta(Z)\phi\omega$$
$$= 2g(Z,\omega)\xi\ 2g(h^*(Z,\omega),\phi X) = 2g(Z,\omega)g(\xi,X)$$
$$+\frac{2}{3}\{(\phi X)\mu - \eta(X)\}g(Z,\omega) \quad (27)$$

The above relation shows that the leaves of $D^{\perp}$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover, the condition $v\mu = 0$, for any $V\ \epsilon\Gamma(D^{\perp})$ implies that the leaves of $D^{\perp}$ are extrinsic spheres in M, that is the integral manifold $N_{\perp}$ of $D^{\perp}$ is ambillical and its mean curvature vector field is non- zero and parallel along N_T. Hence by a result of (11) M is locally a warped product $M = N_T \times_f N_{\perp}$, where N_T and $N_{\perp}$ denote integral manifold of the distribution $D\oplus\langle\xi\rangle$ and $D^{\perp}$, respectively and first the warping function. Thus, the theorem is proved completely.

4. INEQUALITY FOR CONTACT CR-WARPED PRODUCT

We proved the following main result in this section of a general sharped inequality for the length second fundamental form of warped product submanifold.

Theorem 4.1. Let $M = N_T \times_f N_{\perp}$ be a contact CR-warped product submanifold of a nearly Lorentzian para-Sasakian manifold with a semi symmetric non metric connection $\overline{M}$ such that N_T is an invariant submanifold tangent to ξ and $N_{\perp}$ an anti invariant submanifold of $\overline{M}$. Then, we have

(i) The second fundamental form of M satisfying the inequality

$$\|h\|^2 \geq \frac{2}{9}s\,\|\Delta lnf\|^2, \quad (28)$$

where s is the dimension of $N_{\perp}$ and $\nabla \ln f$ is the gradient of $\ln f$.

(ii) If the equality sign of (28) hold identically then N_T is the totally geodesic submanifold and $N_{\perp}$ is the totally umbilical submanifold of $\overline{M}$. Morever, M is a minimal submanifold in $\overline{M}$.

Proof: Let $\overline{M}$ be a $(2n+1)$-dimantional nearly Lorentzian para-Sasakian manifold with a semi symmetric non metric connectionand $M = N_T \times_f N_{\perp}$ be an m-dimentional contact CR-warped product submanifold of $\overline{M}$. Let us consider $\dim N_T = 2p+1$,and $dimN_{\perp} = s$ then $m = 2p+1+s$. Let $\{e_1 \ldots, e_p; \phi e_1 = e_{p+1}, \ldots, \phi e_p = e_{2p}, e_{2p+1} = \xi\}$ and $\{e_{(2p+1)+1}, \ldots\ldots, e_m\}$ be the local orthogonal frames on N_T, and $N_{\perp}$ respectively. Then the orthogonal frames in the normal bundle $T^{\perp}M$of $\phi D^{\perp}$ and $v\{\phi e_{(2p+1)+1}, \ldots, \phi e_m\}$ and $\{e_{m+s+1}, \ldots\ldots, e_{2n+1}\}$ respectively. Then the length of second fundamental form h is defined as

$$\|h\|^2 = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2$$

For the assumed frames, the above equation can be written as

$$\|h\|^2 = \sum_{r=m+1}^{m+s}\sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2$$
$$+\sum_{r=m+s+1}^{2n+1}\sum_{i,j=1}^{m} g\big(h(e_i,e_j), e_r\big)^2. \quad (29)$$

The first term in the right hand side of the above equality is the $\phi D^{\perp}$-component and the term is v-component. If we have only the $\phi D^{\perp}$-component, then we have

$$\|h\|^2 \geq \sum_{r=m+1}^{m+s}\sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2$$

For the given frame of $\phi D^{\perp}$, the above equation will be

$$\|h\|^2 \geq \sum_{k=(2p+1)+1}^{m}\sum_{i,j=1}^{m} g(h(e_i, e_j), \phi e_k)^2$$

Let us decompose the above the equation in terms of the component of $h(D,D), h(D,D^{\perp})$ and $h(D^{\perp},D^{\perp})$ then we have

$$\|h\|^2 \geq \sum_{k=2p+2}^{m}\sum_{i,j=1}^{2p+1} g\big(h(e_i,e_j),\varphi e_k\big)^2$$
$$+2\sum_{k=2p+2}^{m}\sum_{i=1}^{2p+1}\sum_{j=2p+2}^{m} g\big(h(e_i,e_j),\phi e_k\big)^2$$
$$+\sum_{k=2p+2}^{m}\sum_{i,j=2p+2}^{m} g\big(h(e_i,e_j),\phi e_k\big)^2 \quad (30)$$

By Lemma 3.4 (ii), the first term of the right hand side of (30) is identically zero and we shall compute the next term and will left the last term

$$\|h\|^2 \geq 2\sum_{k=2p+2}^{m}\sum_{i=1}^{2p+1}\sum_{j=2p+2}^{m} g\big(h(e_i,e_j),\phi e_k\big)^2$$

As $j,k = 2p+2, \ldots\ldots, m$ then the above equation can be written for one summation as

$$\|h\|^2 \geq 2\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m} g\big(h(e_i,e_j),\phi e_k\big)^2 \quad (31)$$
$$g(h(X,Z),\phi\omega) = \frac{1}{3}[(\phi X lnf)g(Z,\omega) - \eta(X)g(Z,\omega) \quad (32)$$
$$\|h\|^2 \geq 2\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}\Big[\frac{1}{3}(\phi e_i lnf)\,g\big(e_j,e_k\big)^2$$
$$-g\big(e_j,e_k\big)\eta(e_k)\Big]^2 \quad (33)$$
$$\|h\|^2 \geq 2\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}\frac{1}{9}(\phi e_i lnf)^2 g\big(e_j, e_k\big)^2$$
$$+\frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}\eta(e_i)^2\, g\big(e_j,e_k\big)^2$$
$$-\frac{4}{3}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i\ lnf)\eta(e_i)g(e_j, e_k)g(e_j, e_k) \quad (34)$$
$$\|h\|^2 \geq \frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2 g\big(e_j, e_k\big)^2 + \frac{2}{9}s \quad (35)$$

On the other hand from (10)

$$\|\nabla lnf\|^2 = \sum_{i=1}^{n}(e_i(f))^2$$
$$\|\nabla lnf\|^2 = \sum_{i=1}^{n}(e_i(f))^2 + \sum_{i=1}^{n}(\phi e_i lnf)^2 + (\xi lnf)^2 \quad (36)$$
$$\|h\|^2 \geq \frac{2}{9}s + \frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2 g\big(e_j, e_k\big)^2$$
$$+\frac{2}{9}\sum_{j,k=2p+2}^{m}(\xi lnf)^2 g(e_j,e_k)^2$$
$$-\frac{2}{9}\sum_{j,k=2p+1}^{m}(\xi lnf)^2\, g\big(e_j, e_k\big)^2$$
$$\geq \frac{2}{9}s + \frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2 g\big(e_j,e_k\big)^2$$
$$\geq \frac{2}{9}s + \frac{2}{9}\sum_{i=1}^{2p}\sum_{j,k=2p+1}^{m}(\phi e_i lnf)^2 g\big(e_j, e_k\big)^2$$
$$+\frac{2}{9}\sum_{j,k=2p+1}^{m}(\xi lnf)^2 g(e_j e_k)^2$$
$$-\frac{2}{9}\sum_{j,k=2p+1}^{m}(\xi lnf)\ g(e_j,e_k)^2$$
$$\geq \frac{2}{s}s - \frac{2}{9}\sum_{j,k=2p+2}^{m}(\xi lnf)^2 g(e_j,e_k)^2$$
$$+\frac{2}{9}\sum_{i=1}^{2p+1}\sum_{j,k=2p+2}^{m}(\phi e_i lnf)^2 g\big(e_j, e_k\big)^2$$
$$+\frac{2}{9}\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}\big(e_i lnf\big)^2\, g(e_j,e_k)^2$$

$+\frac{2}{9}\sum_{j,k=2p+2}^{m}(\xi lnf)\, g(e_j, e_k)^2$
$\geq \frac{2}{9}s - \frac{2}{9}s + \frac{2}{9}s\|\nabla lnf\|^2$
$\Rightarrow \|h\|^2 \geq \frac{2}{9}s\, \|\nabla lnf\|^2$

which is the inequality (28). Let h be the second fundamental form of $N_\perp$ in M, then from (26), we have

$$h^*(Z,\omega) = g(Z,\omega)\nabla \ln f \quad (37)$$

For any $Z, W\epsilon\Gamma(D^\perp)$.Now assume that the equality case of (28) holds identically. Then from (29), (30) and (31), we obtain

$$h(D,D) = 0, \quad h(D,D^\perp) \subset \phi D^\perp. \quad (38)$$

Since N_T is a totally geodesic submanifold in M (by Lemma 2.1 (i), using this fact with the first condition in (38) implies that N_T is totally geodesic in $\bar{M}$. on the other hand, by direct calculation same as in the proof of Theorem 3.4, we deduce that $N_\perp$ is totally umbilical in M. therefore, the second condition of (38) with (37) implies that $N_\perp$ is totally umbilical in $\bar{M}$. Moreover, all three conditions of (38) imply that M is minimal submanifold of $\bar{M}$. This complete the proof of the theorem.

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