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INVERSE THEOREM BY BERNSTEIN-NEWTON INTERPOLATION POLYNOMIALS

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ABSTRACT: The Bernstein-Newton interpolation polynomials are defined for $f \epsilon L_p[0,1]$ as

$$L_{n,m}(\mathrm{f,t}) = (n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(\mathrm{t})\int_0^1 p_{n\vartheta}(\mathrm{u})\left\{f(u) + \sum_{j=1}^{m}\frac{n^{\frac{j}{2}}}{j!}\prod_{i=0}^{j-1}\left(t-\left(u+\frac{i}{\sqrt{n}}\right)\right)\Delta^j f(u)\right\}du.$$

In this paper we have tried to prove $\omega_{m+1}(f,\tau,p,I) = O(\tau^{\propto})$, where $\propto < m+1$.

KEYWORDS: Newton-interpolation polynomial, Durrmeyer modification of operators, Bernstein polynomials

1. INTRODUCTION[1]

Bernstein polynomials ([1], [2], [5], [9] & [12]) show an affinity for convergence to continuous functions. The convergence for integrable functions in L_p-norm has been studied in [9]. The limitation in convergence has been overcome by different methods in ([4], [7] & [12]). However, Newton-Bernstein polynomials have exhibited better rate of convergence in [10] & [11]. Research workers have investigated the smoothness of function, given the proximity of linear (positive) operators to function. This is also known as Inverse theorem. In this paper we have analyzed the nature of function from the given rate of convergence.

2. DEFINITION AND AUXILIARY RESULTS

Bernstein-Newton-interpolation polynomials are defined for $f \,\epsilon\, L_p(I), 1 \le p < \infty$ as

$$L_{n,m}(f,t) = (n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\int_0^1 p_{n\vartheta}(u)p_m(u,t)du, \qquad (2.01)$$

where

$$p_m(u,t) = \sum_{j=0}^{m}\frac{n^{j/2}}{j!}\prod_{i=0}^{j-1}\left(t-\left(u+\frac{i}{\sqrt{n}}\right)\right)\Delta^j f(u), \qquad (2.02)$$

$p_{n\vartheta}(t) = \binom{n}{\vartheta}t^{\vartheta}(1-t)^{n-\vartheta}$, Δ is forward difference of step size $\frac{1}{\sqrt{n}}$. It follows from Beta integral that

$(n+1)\int_0^1 p_{n\vartheta}(u)du = 1$

and hence

$$L_{n,m}(1,t) = 1 \qquad (2.03)$$

It follows from interpolation properties [3] that

$$L_{n,m}(u^r,t) = t^r; r \le m \qquad (2.04)$$

and hence

$$L_{n,m}((u-t)^r,t) = 0; r \le m. \qquad (2.05)$$

Notations: $I = [0,1], I_1[a_1,b_1], I_2 = [a_2,b_2]$;

$0 < a_1 < a_2 < b_2 < b_1 < 1$.

Lemma 2.1: There holds for $r \,\epsilon\, N$ and a positive real number 's'

(i) $(n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\int_0^1 p_{n\vartheta}(u)|u-t|^r du \le \frac{c}{(\sqrt{n})^r}$,

(ii) $(n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\int_0^1 p_{n\vartheta}(u)|u-t|^s du \le \frac{c}{(\sqrt{n})^s}$,

(iii) $(n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)|\vartheta - nt|^r\int_0^1 p_{n\vartheta}(u)|u-t|^s du \le \frac{c}{(\sqrt{n})^{s-r}}$,

Uniformly in $t \,\epsilon\, [0,1]$.

Proof. (i) The proof follows from properties of Beta integral & [10] .

(ii) Let r be a natural number > s. Let $p_1 = \frac{r}{s}$.Then $p_1 > 1$. Let q_1 be its conjugate. Then last inequality now follows from Holder's inequality & (1)

$$\left|(n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\int_0^1 p_{n\vartheta}(u)|u-t|^s du\right|$$

$$\le \left(\sum_{\vartheta}^{n} p_{n\vartheta}(t)\right)^{1/q_1}\left(\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\left(\int_0^1 (n+1)p_{n\vartheta}(u)|u-t|^s\right)^{p_1}\right)^{1/p_1}$$

$$\le \sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\left(\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\int_0^1 (n+1)p_{n\vartheta}(u)|u-t|^{sp_1}du\right)^{1/p_1} \le \frac{c}{(\sqrt{n})^s}, \text{ from (i).}$$

(iii) An application of Holder's inequality in sum followed by Holder's inequality in integral entails

$$(n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)|\vartheta - nt|^r\int_0^1 p_{n\vartheta}(u)|u-t|^s du$$

$$\le \left(\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)|\vartheta\text{-}nt|^{rp}\right)^{1/p}\left(\sum_{\vartheta=0}^{n} p_{n\vartheta}\left(\int_0^1 (n+1)p_{n\vartheta}(u)|u\text{-}t|^s du\right)^q\right)^{1/q}$$

$$\le c_1(\sqrt{n})^r\left\{\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\int_0^1 p_{n\vartheta}(u)|u-t|^{sq}du\right\}^{1/q}$$

$$\le \frac{c_2}{(\sqrt{n})^{s-r}},$$

using order of moments of Bernstein polynomials [6] and (ii).

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Lemma 2.2 ([10] & [11]): Let $f \in L_p(I), p > 1$ and $s \in R^+$. Then there holds
$\|L_{n,m}(f(u)(u-t)^s, t)\|_{L_p(I_2)} \le \frac{c}{(\sqrt{n})^s}\|f\|_{L_p(I_1)} + \frac{c}{n^l}\|f\|_{L_p(I)}$,
where l > 0 is arbitrary.

Corollary 2.3 ([10] & [11]): Let $f \in L_p(I)$, $p > 1$, $r \in N$, $s \in R^+$. Then
$\left\|(n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)|\vartheta-nt|^r \int_0^1 p_{n\vartheta}(u)|u-t|^s f(u)\,du\right\| \le \frac{c}{(\sqrt{n})^{s-r}}\|f\|_{L_p(I)}$.

Proof. An application of Holder's inequality in sum followed by Holder's Inequality in integral yields
$\left|(n+1)\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)|\vartheta-nt|^r \int_0^1 p_{n\vartheta}(u)|u-t|^s f(u)\,du\right|^p$
$\le \left(\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)|\vartheta-nt|^{rq}\right)^{1/q}$
$\times \left(\sum_{\vartheta=0}^{n} p_{n\vartheta}(t)\left(\int_0^1 (n+1)p_{n\vartheta}(u)|u-t|^s |f(u)|\right)^p\right)$
$\le c(\sqrt{n})^r \frac{1}{(\sqrt{n})^s}\|f\|_{L_p(I)}$.
This follows from order of moment of Bernstein polynomials [6] and Lemma 2.2.

Lemma 2.2 ([10] & [11]): Let $f \in L_p(I)$, $p > 1$, $r \in N$. Then there holds
$\left\|L_{n,m}\left((u-t)^r \int_t^u f(w)dw, t\right)\right\|_{L_p(I_2)} \le \frac{c}{(\sqrt{n})^{r+1}}\|f\|_{L_p(I_1)} + \frac{1}{n^l}\|f\|_{L_p(I)}$,
where l >0 is arbitrary.

Lemma 2.5 ([6]) Let $\omega(\tau)$ be a monotonic increasing function of τ.Let $0 < \alpha < r$ and τ, η be small positive numbers such that $\omega(\tau) \le c\left\{\eta^\alpha + \left(\frac{\tau}{\eta}\right)^r \omega(\eta)\right\}$. Then $\omega(\tau) = O(\tau^\alpha)$, as $\tau \to 0$.

Lemma 2.6 ([9], p.107): Let $1 \le p < \infty$ and $f \in L^p[a,b]$.If f has m derivatives on $[a,b]$, where $f^{(m-1)} \in A.C.[a,b]$ and $f^{(m)} \in L^p[a,b]$, then
$\Delta_\delta^m f(x) = \int_0^\delta \dots \int_0^\delta f^{(m)}(x + \sum_{i=1}^m t_i)\,dt_1 \dots dt_m$;
$x \in [a, b - m\delta]$.
Let $1 \le p < \infty$, $f \in L^p[a,b]$ and $[a_1,b_1] \subset [a,b]$. The Steklov mean $f_{n,m}(t)$ of m^{th} order for sufficiently small $\eta > 0$ is defined as
$f_{\eta,m}(t) =$
$\eta^{-m}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \dots \int_{\frac{-\eta}{2}}^{\frac{\eta}{2}} \left(f(t) + (-1)^{m-1}\Delta_{\sum_{i=1}^m t_i}^m f(t)\right) dt_1 \dots dt_m$,
$t \in [a_1, b_1]$. The integral modulus of smoothness for $f \in L^p[a,b]$ is defined as
$\omega_m(f,\tau,p,[a,b]) = Sup_{0<\delta\le\tau}\|\Delta_\delta^m f(t)\|_{L^p[a,b-m\delta]}$.
We obtain estimates for derivatives of Steklov means in terms of corresponding order of integral modulus of smoothness of the function. Some of the elementary values have been discussed in Ditzian and May ([2] & [7]) when $f \in L^p$ is periodic, proof is given in Timan ([9], p.167).

Lemma 2.7: The function $f_{\eta,m}$ has derivatives up to order m, the $(m-1)^{\text{th}}$ derivative is absolutely continuous on $[a_1,b_1]$ and m^{th} derivative exists a.e. and belongs to $L^p[a_1,b_1]$. Moreover, there holds
(1) $\|f_{\eta,m}^{(r)}\|_{L^p[a_1,b_1]} \le c_r \eta^{-r}\omega_r(f,\eta,p,[a,b]), r = 1,2,3, \dots .m;$
(2) $\|f - f_{\eta,m}\|_{L^p[a_1,b_1]} \le c_{m+1}\omega_m(f,\eta,p,[a,b])$,
(3) $\|f_{\eta,m}\|_{L^p[a_1,b_1]} \le c_{m+2}\|f\|_{L^p[a,b]}$,
(4) $\|f_{\eta,m}^{(m)}\|_{L^p[a_1,b_1]} \le c_{m+3}\eta^{-m}\|f\|_{L^p[a,b]}$,
where c_j's are constants independent of f and η.

2. 3 MAIN RESULTS

Theorem 3.1: Let $1 < p < \infty$ and $f \in L_p(I)$ satisfy
$\|L_{n,m}(f,t) - f(t)\|_{L_p(I_1)} = O\left(\frac{1}{\sqrt{n}}\right)^\alpha$,as $n \to \infty$.
Then $\omega_{m+1}(f,\tau,p,I_2) = O(\tau^\alpha)$, as $\tau \to 0$, where $0 < \alpha < (m+1)$.
In the proof of this theorem we require two lemmas which we prove first.

Lemma 3.2: Let $1 < p < \infty$ and $f \in L^p$(I) with support $f \subset I_2$. Then
$\left\|\frac{d^{m+1}}{dt^{m+1}}L_{n,m}(f,t)\right\|_{L_p(I_2)} \le c(\sqrt{n})^{m+1}\|f\|_{L_p(I_2)}$.

Proof. A typical term of $L_{n,m}(f,t)$ is
$$T_1(t) = c_1(n+1)(\sqrt{n})^{j-r_2}\left\{\sum_{\vartheta=0}^{n}\binom{n}{\vartheta}\int_0^1\{t^\vartheta(1-t)^{n-\vartheta}(t- ur1\times pn\vartheta ufu+r3ndu \qquad (3.01)$$
where $r_1 + r_2 = j$, $r_1 > 0$; $0 \le r_3 \le$j. we find from (Lemma p. 26 [6]).
That a typical component of $\frac{d^{m+1}}{dt^{m+1}}(T_1(t))$ is
$$T_{11}(t) = c_1(n+1)(\sqrt{n})^{j-r_2} n^i \sum_{\vartheta=0}^{n} p_{n\vartheta}(t)(\vartheta - nt)^{j_o} \times \int_0^1 p_{n\vartheta}(u) f\left(u + \frac{r_3}{\sqrt{n}}\right)(t-u)^{r_1-s}du, \qquad (3.02)$$
where $2i + j_o \le k$, $0 \le k \le m+1$, $s = m+1-k; i, j_o \ge 0$.
This is bounded in L_p-norm;by Corollary 2.3,as
$$\|T_{11}(t)\|_{L_p(I_2)} \le c_2\frac{(\sqrt{n})^{j-r_2+2i}}{(\sqrt{n})^{r1-s-j_o}}\|f\|_{L_p(I_2)} \le c_2(\sqrt{n})^{m+1}\|f\|_{L_p(I_2)}.$$
This completes proof of Lemma.

Lemma 3.3: Let $1 < p < \infty$, $f \in L_p(I)$ with support $f \subset I_2$. Moreover f has $m+1$ derivatives on I_2 with $f^{(m)} \in A.C.(I_2)$ and $f^{(m+1)} \in L_p(I_2)$, then
$$\left\|\frac{d^{m+1}}{dt^{m+1}}L_{n,m}(f,t)\right\|_{L_p(I_2)} \le c\|f^{(m+1)}\|_{L_p(I_2)}.$$
Proof. The smoothness conditions of f imply that

$$f(u)=\sum_{i=0}^{m}\frac{(u-t)^i}{i!}f^i(t)+\frac{1}{m!}\int_t^u(u-w)^m f^{(m+1)}(w)dw. \quad (3.03)$$

We apply operator $L_{n,m}(.,x)$ on both sides and using (2.3)

$$L_{n,m}(f,x)-f(t)=\frac{1}{m!}L_{n,m}\left(\int_t^u(u-w)^m f^{(m+1)}(w)dw\,,x\right)+\sum_{i=1}^{m}\frac{f^{(i)}(t)}{i!}L_{n,m}((u-t)^i,x).$$

We now use (3.03) in $L_{n,m}(f,x)$ as

$$L_{n,m}(f,x)=(n+1)\sum_{\vartheta=0}^{x}p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)\left\{\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\prod_{i=0}^{j-1}\left(x-u-\frac{i}{\sqrt{n}}\right)\Delta^j\right.$$
$$\left.\times\left\{\sum_{r=0}^{m}\frac{(u-t)^r}{r!}f^{(r)}(t)+\frac{1}{m!}\int_t^u(u-w)^m f^{(m+1)}(w)dw\right\}\right\}du \quad (3.04)$$

We put $(u-t)^r=(\overline{u-x}+\overline{x-t})^r$,

$$\sum_{r_4=0}^{r}\binom{r}{r_4}(u-x)^{r_4}(x-t)^{r-r_4}. \quad (3.05)$$

From (3.04)

$$L_{n,m}(f,x)=(n+1)\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)$$
$$\times\int_0^1 p_{n\vartheta}(u)\left\{\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\prod_{i=0}^{j-1}\left(x-u-\frac{i}{\sqrt{n}}\right)\right\}\Delta^j$$
$$\times\left\{\sum_{r=0}^{m}\frac{(u-t)^r}{r!}f^{(r)}(t)\right\}du+\frac{1}{m!}(n+1)\sum_{\vartheta=0}^{m}p_{\vartheta n}(x)$$
$$\times\int_0^1 p_{n\vartheta}(u)\left\{\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\prod_{i=0}^{j-1}\left(x-u-\frac{i}{\sqrt{n}}\right)\times\Delta^j\left(\int_t^u(u-\right.\right.$$

wmfm+1wdwdu

$$=\sum_{r=0}^{m}T_{2,r}+T_{2,m+1}\,, \quad (3.06)$$

$$T_{2,0}=(n+1)\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)f(t)du=f(t) \quad (3.07)$$

In $T_{2,r}$,,we use (3.05) to obtain (r+1)terms. When $r_4=0$, this term

$$=(n+1)\sum p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)\times\left\{\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\prod_{i=0}^{j-1}\left(x-u-\frac{i}{\sqrt{n}}\right)\Delta^j\left\{\frac{f^{(r)}(t)}{r!}(x-t)^r\right\}\right\}du$$
$$=\frac{f^{(r)}(t)}{r!}(x-t)^r, \quad (3.08)$$

when $r_4\neq 0$;

$$T_{6,r}=\sum_{r_4=1}^{r}\binom{r}{r_4}(x-t)^{r-r_4}\,(n+1)\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)$$
$$\times\left\{\sum_{j=0}^{m}\frac{n^{j/2}}{j!}\prod_{i=0}^{j-1}\left(x-u-\frac{i}{\sqrt{n}}\right)\Delta^j\left\{\frac{f^{(r)}(t)}{r!}(u-x)^{r_4}\right\}\right\}du=0. \quad (3.09)$$

A general component of $T_{2,m+1}$ is

$$\Xi(x,t)=c_1(n+1)\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)(\sqrt{n})^{j-r_2}(x-u)^{r_1}$$
$$\times\int_t^{u+\frac{r_3}{\sqrt{n}}}\left(u-\omega+\frac{r_3}{\sqrt{n}}\right)^m f^{(m+1)}(w)dwdu \quad (3.10)$$

$$=c_1(n+1)\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)(\sqrt{n})^{j-r_2}(x-u)^{r_1}$$
$$\times\int_t^{u}\left(u-w+\frac{r_3}{\sqrt{n}}\right)^m f^{(m+1)}(w)dwdu+c_1(n+1)$$
$$\times\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)(\sqrt{n})^{j-r_2}(x-u)^{r_1}$$
$$\times\int_u^{u+\frac{r_3}{\sqrt{n}}}\left(u-w+\frac{r_3}{\sqrt{n}}\right)^m f^{(m+1)}(w)dwdu$$
$$=c_2(n+1)\sum_{r_5=0}^{m}(\sqrt{n})^{j-r_2-m+r_5}\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)\times\int_0^1 p_{n\vartheta}(u)(x-u)^{r_1}\int_t^u(u-w)^{r_5}f^{(m+1)}(w)dwdu$$
$$+c_2(n+1)\sum_{r_5=0}^{m}(\sqrt{n})^{j-r_2-m+r_5}\sum_{\vartheta=0}^{n}p_{n\vartheta}(x)\int_0^1 p_{n\vartheta}(u)(x-u)^{r_1}$$
$$\times\int_t^{u+\frac{r_3}{\sqrt{n}}}(u-w)^{r_5}f^{(m+1)}(w)\,dw\,du$$
$$=c_2(n+1)\sum_{r_5=0}^{m}(\sqrt{n})^{r_6}(\xi_1(x,t)+\xi_2(x,t)),$$

say $(r_6=j-r_2-m+r_5)$ (3.11)

Now, $\frac{d^{m+1}}{dx^{m+1}}\xi_1(x,t)$ consists of finite number of terms of type as

$$c_3\sum_{\vartheta=0}^{n}(n+1)n^i p_{n\vartheta}(u)(\vartheta-nx)^{j_0}\quad\int_0^1 p_{n\vartheta}(u)(x-u)^{r_1-s}\times\int_t^u(u-w)^{r_5}f^{(m+1)}(w)dwdu, \quad (3.12)$$

where $i,j_o\geq 0;2i+j_0\leq k,s=m+1-k,\ 0\leq k\leq m+1;\ 0\leq r_5\leq m$. Thus

$$\left|\frac{d^{m+1}}{dx^{m+1}}\xi_1(x,t)\right|_{at\ x=t}\leq c_3\sum_{i,j}n^i\sum p_{n\vartheta}(t)|\vartheta-nt|^{j_0}(n+1)\times\int_0^1 p_{n\vartheta}(u)|u-t|^{r_1-s+r_5}\left|\int_t^u\left|f^{(m+1)}(w)\right|dw\right|du,$$

Making a use of Holder's inequality in sum and in integeral (twice) and properties of Hardy Little-wood majorant ([8],P.05)

$$\sum_{\vartheta=0}^{n}p_{n\vartheta}(t)|\vartheta\text{-}nt|^{j_0}\int_0^1(n+1)p_{n\vartheta}(u)|u\text{-}t|^{r_1-s+r_5+1}\times\left|\frac{1}{|u-t|}\int_t^u f^{(m+1)}(w)dw\right|du$$
$$\leq\left(\sum_{\vartheta=0}^{n}p_{n\vartheta}(t)|\vartheta-nt|^{j_0q}\right)^{1/q}\times\left\{\sum_{\vartheta=0}^{n}p_{n\vartheta}(t)\left\{\int_0^1(n+\right.\right.$$

1pnϑ(u)u−tr1−s+r5+1Hfm+1(u)dup1p

$$\leq c_4(\sqrt{n})^j\left\{\sum_{\vartheta=0}^{n}p_{n\vartheta}(t)\int_0^1(n+1)p_{n\vartheta}(u)|u\text{-}t|^{(r_1-s+r_5+1)p}\left|H_{f^{(m+1)}}(u)\right|^p du\right\}^{1/p} \quad (3.13)$$

We now proceed as in the analysis of direct convergence ([10] & [11]) to obtain

$$\left\|\frac{d^{m+1}}{dx^{m+1}}\xi_1(x,t)\right|_{at\ x=t}\Big\|_{L_p(I_2)}\leq c_5 n^{\frac{k}{2}}\frac{1}{(\sqrt{n})^{r_1-s+r_5+1}}\left\|f^{(m+1)}\right\|_{L_p(I_2)}$$
$$=c_5\frac{1}{(\sqrt{n})^{r_1+r_5-m}}\left\|f^{(m+1)}\right\|_{L_p(I_2)}. \quad (3.14)$$

We deal with similarly $\xi_2(x,t)$ as in $\xi_1(x,t)$ and analysing as in the direct convergence ([10]& [11]

$$\left\|\frac{d^{m+1}}{dx^{m+1}}\xi_2(x,t)\right|_{at\ x=t}\Big\|_{L_p(I_2)}\leq\frac{c_6}{(\sqrt{n})^{r_1-m+r_5}}\left\|f^{(m+1)}\right\|_{L_p(I_2)} \quad (3.15)$$

From (3.11), (3.14) *and* (3.15),

$$\left\|\frac{d^{m+1}}{dx^{m+1}}\xi(x,t)\right|_{at\ x=t}\Big\|_{L_p(I_2)}\leq c_7(\sqrt{n})^{j-r_2-m+r_5}\frac{1}{(\sqrt{n})^{r_1-m+r_5}}\left\|f^{(m+1)}\right\|_{L_p(I_2)}$$

$= c_7\|f^{(m+1)}\|_{L_p(I_2)}$

and therefore $\left\|\frac{d^{m+1}}{dt^{m+1}} T_{6,m+1}\right\|_{L_p(I_2)} \leq c_8\|f^{(m+1)}\|_{L_p(I_2)}$

We notice from (3.07), (3.08) and (3.09) that $T_{2,r}$ are polynomials in x of degree $r \leq m$, and as such $\frac{d^{m+1}}{dx^{m+1}} T_{2,r} = 0$. We conclude that

$\left\|\frac{d^{m+1}}{dt^{m+1}} L_{n,m}(f,t)\right\|_{L_p(I_2)} = \left\|\frac{d^{m+1}}{dx^{m+1}} L_{n,m}(f,x)\Big|_{at\ x=t}\right\|$

$\leq c\|f^{(m+1)}\|_{L_p(I_2)}$.

This completes proof of Lemma. We are now in a position to prove our theorem. We choose points $x_i, y_i; i = 1,2,3$ as follows such that $a_1 < x_1 < x_2 < x_3 < a_2$, $a_2 < b_2$, $b_2 < y_3 < y_2 < y_1 < b_1$.

We further choose a function $g \in C_0^{m+1}$ with support $g \subset (x_2, y_2)$ and $g \equiv 1$ on $[x_3, y_3]$. Now,

$$\|\Delta_\gamma^{m+1}(fg)(t)\|_{L_p[x_2,y_2]} \leq \|\Delta_\gamma^{m+1}(fg)(t) - L_{n,m}(fg,t)\|_{L_p[x_2,y_2]} + \|\Delta_\gamma^{m+1} L_{n,m}(fg,t)\|_{L_p[x_2,y_2]} \quad (3.16)$$

Let $\bar{f}_{n,m+1}$ be Steklov mean for $fg = \bar{f}$. We write

$$\Delta_\gamma^{m+1} L_{n,m}(fg,t) = \int_0^\gamma \dots \int_0^\gamma L_{n,m}^{(m+1)}(fg, t + \textstyle\sum_{i=1}^{m+1} t_i)dt_i \dots dt_{m+1}.$$

We use Holder's inequality (m+1) times to obtain

$$|\Delta_\gamma^{m+1} L_{n,m}(fg,t)|^p \leq \gamma^{(m+1)\frac{p}{q}} \int_0^\gamma \dots \int_0^\gamma \left|L_{n,m}^{(m+1)}(fg, t + \textstyle\sum_{i=1}^{m+1} t_i)\right|^p dt_1 \dots dt_{m+1}.$$

We next use Fubini's theorem (m+1)times to obtain

$$\int_{x_2}^{y_2} |\Delta_\gamma^{m+1} L_{n,m}(fg,t)|^p dt \leq \gamma^{(m+1)\frac{p}{q}} \int_0^\gamma \dots \int_0^\gamma \int_{x_2}^{y_2} \left|L_{n,m}^{(m+1)}(fg, t + \textstyle\sum_{i=1}^{m+1} t_i)\right|^p dt_1 \dots dt_{m+1} \quad (3.17)$$

$$\leq \gamma^{(m+1)p} \|L_{n,m}^{(m+1)}(fg,t)\|_{L_p[x_2', y_2']}^p.$$

(for sufficiently small γ) We unite (3.16) and (3.17).Thus

$$\|\Delta_\gamma^{m+1} L_{n,m}(fg,t)\|_{L_p[x_2,y_2]} \leq \left\|\Delta_\gamma^{m+1}\left((fg)(t) - L_{n,m}(fg,t)\right)\right\|_{L_p[x_2,y_2]}$$

$$+\gamma^{(m+1)} \|L_{n,m}^{(m+1)}(fg,t)\|_{L_p[x_2', y_2']},$$

where $x_2' = x_2$,

$y_2' = y_2 + (m+1)\gamma \leq \|\Delta_\gamma^{m+1}\{\bar{f}(t) - L_{n,m}(\bar{f},t)\}\|_{L_p[x_2',y_2']} + \gamma^{m+1}$

$\times \left\{\|L_{n,m}^{(m+1)}(\bar{f} - \bar{f}_{\eta,m+1}, t)\|_{L_p[x_2',y_2']} + \|L_{n,m}^{(m+1)}(\bar{f}_{\eta,m+1}, t)\|_{L_p[x_2',y_2']}\right\}$

$\leq \|\Delta_\gamma^{m+1}\{\bar{f}(t) - L_{n,m}(\bar{f},t)\}\|_{L_p[x_2',y_2']} + c_1\gamma^{m+1}$

$\times \left\{(\sqrt{n})^{m+1}\|\bar{f} - \bar{f}_{\eta,m+1}\|_{L_p[x_2,y_2]} + \|\bar{f}^{(m+1)}_{,\eta,m+1}\|_{L_p[x_2,y_2]}\right\}$

(by Lemma (3.02) and (3.03)

$\leq \|\Delta_\gamma^{m+1}\{\bar{f}(t) - L_{n,m}(\bar{f},t)\}\|_{L_p[x_2,y_2]}$

$$+c_9\gamma^{m+1}\left((\sqrt{n})^{m+1} + \frac{1}{\eta^{m+1}}\right) \times \omega_{m+1}(\bar{f}, \eta, p, [x_2, y_2]) \quad (3.18)$$

(by Lemma 2.6)

Our main task is to prove that

$$\|\Delta_\gamma^{m+1}\{\bar{f}(t) - L_{n,m}(\bar{f},t)\}\|_{L_p[x_2,y_2]} = O\left(\frac{1}{n^{\alpha/2}}\right) \quad (3.19)$$

Once (3.18) is proved, we have from (3.18) and (3.19)

$\|\Delta_\gamma^{m+1}(fg)(t)\|_{L_p[x_2,y_2]}$

$$\leq c_{10}\left\{\frac{1}{n^{\alpha/2}} + \gamma^{m1}\left((\sqrt{n})^{m+1} + \frac{1}{\eta^{m+1}}\right)\omega_{m+1}(\bar{f}, \eta, p, [x_2, y_2])\right\}.$$

For sufficiently small values of γ *and* η. we choose $\eta = \frac{1}{\sqrt{n}}$ and taking sup of $\gamma \leq \tau$, we obtain

$\omega_{m+1}(\bar{f}, \tau, p, [x_2, y_2]]$

$\leq c_{11}\left\{\eta^\alpha + \left(\frac{\tau}{\eta}\right)^{m+1} \omega_{m+1}(f, \eta, p, [x_2, y_2]\right\}$.

This implies by Lemma 2.5 that

$\omega_{m+1}(fg, \tau, p, [x_2 y_2]) = O(\tau^\alpha)$;as $\tau \to 0$,

This further implies that

$$\omega_{m+1}(f, \tau, p, I_2) = O(\tau^\alpha)\ ; as\ \tau \to 0. \quad (3.20)$$

We accomplish this by induction on α. Let $0 < \alpha \leq 1$,

$f(t)g(t) - L_{n,m}(f(u)g(u), t)$

$= g(t)\{f(t) - L_{n,m}(f,t)\} - L_{n,m}(f(u)(g(u) - g(t),t))$

$$= g(t)\left(f(t) - L_{n,m}f(t)\right) - L_{n,m}(f(u)(u-t)g'(\xi), t), \quad (3.21)$$

where ξ lies between u and t

It follows from hypothesis of theorem and Lemma 2.2 that right hand side of (3.20) is $O\left(\frac{1}{n^{\alpha/2}}\right) + O\left(\frac{1}{n^{1/2}}\right) = O\left(\frac{1}{n^{\alpha/2}}\right)$, $n \to \infty$. We now assume that (3.16) is true for $r - 1 \leq \alpha < r$, $\alpha = (r-1) + \theta$, $0 \leq \theta < 1$, and prove (3.19) true for $r \leq \alpha < r + 1$. Let $f_{\eta,m+1}$ be steklov mean for f

$f(t)g(t) - L_{n,m}(f(u)g(u), t)$

$= g(t)\{f(t) - L_{n,m}(f(u), t)\} - L_{n,m}(f(u)(g(u) - g(t), t))$.

We expand $f(u)(g(u) - g(t))$ as

$$S_1(u,t) = f(u)\left\{(u-t)g'(t) + \frac{(u-t)^2}{2!}g^{(2)}(t) + \cdots\right\} \quad (3.22)$$

Taking the first term in $S_1(u,t)$

$L_{n,m}(f(u)(u-t)g'(t), t)$

$= g'(t)\left\{L_{n,m}\left(f(u) - f_\eta(u)\right)(u-t), t\right.$

$\left. + L_{n,m}\left((f_\eta(u) - f_\eta(t))(u-t), t)\right) + f_\eta(t)L_{n,m}(u-t), t\right\}$

$$= (T_3(t) + T_4(t) + T_5(t))\ g'(t). \quad (3.23)$$

By lemma 2.2

$\|T_3(t)\|_{L_p[x_2,y_2]} \leq c_{12}\left\{\frac{1}{\sqrt{n}}\|f - f_\eta\|_{L_p[x_1,y_1]} \frac{1}{n^l}\|f\|_{L_p[I]}\right\}$.

We use (2.5) to obtain

$$T_4(t) = L_{n,m}\left((u-t)\left\{\sum_{i=1}^m \frac{(u-t)^i}{i!} f_\eta^{(i)}(t) + \frac{1}{m!}\int_t^u (u-w)^m f_\eta^{(m+1)}(w)dw\right\}, t\right)$$

$$= L_{n,m}\left(\frac{(u-t)^{m+1}}{m!} f_\eta^{(m)}(t), t\right) + \frac{1}{m!} L_{n,m}\left((u-t)\int_t^u (u-w)^m f_\eta^{(m+1)}(w)dw, t\right) \quad (3.24)$$

$= T_{41}(t) + T_{42}(t)$.

It follows from Lemma 2.1 that

$$\|T_{41}(t)\|_{L^p[x_2,y_2]} \leq \frac{c_{13}}{(\sqrt{n})^{m+1}} \|f_\eta^{(m)}\|_{L^p[x_2,y_2]} \quad (3.25)$$

Finally, Lemma 2.4 is applied to $T_{42}(t)$

$\|T_{42}(t)\|_{L^p[x_2,y_2]}$

$$\leq c_{14}\left\{\frac{1}{(\sqrt{n})^{m+2}}\left\|f_\eta^{(m+1)}\right\|_{L^p[x_1,y_1]}+\frac{1}{n^l}\left\|f_\eta^{(m+1)}\right\|_{L^p(I).}\right\}, \quad (3.26)$$

and by (2.5)

$$T_5(t)=0. \quad (3.27)$$

The later terms of $S_1(u,t)$ yield better order. Taking into account (3.20) to (3.27)

$\|L_{n,m}(f(u)(u-t)g'(t),t)\|_{L_p[x_2,y_2]}$

$$\leq c_{15}\left\{\frac{1}{\sqrt{n}}\{\|f-f_\eta\|\}_{L_p[x_1,y_1]}+\frac{1}{n^l}\|f\|_{L_p(I)} + \frac{1}{(\sqrt{n})^{m+1}}\left\|f_\eta^{(m)}\right\|_{L^p[x_2,y_2]}+\frac{1}{(\sqrt{n})^{m+2}}\left\|f_\eta^{(m+1)}\right\|_{L_p[x_1,y_1]} + \frac{1}{n^l}\left\|f_\eta^{(m+1)}\right\|_{L_p[I]}\right\}. \quad (3.28)$$

We recall the properties related to Steklov mean in Lemma 2.7 and obtain

$\|L_{n,m}(f(u)(u-t)g'(t),t)\|_{L_p[x_2,y_2]}$

$$\leq c_{16}\left\{\frac{1}{\sqrt{n}}\omega_{m+1}(f,\eta,p,[x_1',y_1'])+\frac{1}{n^l}\|f\|_{L_p(I)} + \frac{1}{(\sqrt{n})^{m+1}}\eta^{-m}\omega_{m+1}(f,\eta,p,[x_1,y_1] + \frac{1}{(\sqrt{n})^{m+2}}\eta^{-(m+1)}\omega_{m+1}(f,\eta,p,[x_1',y_1'] + \frac{1}{n^l}\eta^{-(m+1)}\|f\|_{L_p(I)}\right\}, \quad (3.29)$$

(where$x_1'=x_1-(m+1)n\,, y_1'=y_1+(m+1)\eta$).

$$\leq c_{17}\left\{\frac{1}{\sqrt{n}}.\eta^{\propto}+\frac{1}{n^l}+\frac{1}{(\sqrt{n})^{m+1}}.\eta^{-m}\omega_m(f,\eta,p,[x_1,y_1])\right. + \frac{1}{(\sqrt{n})^{m+2}}\eta^{-(m+1)}\eta^{\propto}+\frac{1}{n^l}\eta^{-(m+1)}\|f\|_{L_p(I)}. \quad (3.30)$$

We have thus from (3.20) $\omega_{m+1}(f,\eta,p,[x_1,y_1]=O(\tau^{\propto})$ From Lemma ([9] Theorem 6.1.2)

$$\omega_m(f,\eta,p,[x_1,y_1]=O(\eta^{\propto}). \quad (3.31)$$

We choose $\eta=\frac{1}{\sqrt{n}}$ and l large enough, In (3.30) and utilizing (3.31)

$$\|L_{n,m}(f(u)(u-t)g'(t),t)\|_{L_p[x_2,y_2]}\leq c_{17}\frac{1}{(\sqrt{n})^{\propto+1}}.$$

This in conjunction with (3.21) and hypothesis of theorem, proves (3.20). Hence the induction hypothesis is complete. Thus we have completed proof of theorem.

4. CONCLUSION

The modification studied have wide application to other situation as well.

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