Proceedings of the Mathematical Society
Banaras Hindu University, Vol. 34 (2018), pp. 5-9

www.msbhu.in

ISSN 0970-7080
Mathematical Society
B.H.U.

CR-SUBMANIFOLDS OF A NEARLY QUASI SASAKIAN MANIFOLD WITH A CONNECTION

Shamsur Rahman[1]
Department of Mathematics Maulana Azad National Urdu University Polytechnic Satellite Campus
Darbhanga Bihar 846001, India
E-mail: shamsur@rediffmail.com

(Received on May 12, 2019)

ABSTRACT: The present paper deals with different geometrical properties of the *CR*-submanifold of a nearly Quasi Sasakian manifold, generalizing the results of a nearly Quasi Sasakian manifold and thus those of Sasakian manifolds with the quarter-symmetric metric connection. In the end, we studied the properties of the parallel distribution relating to ξ-vertical CR-submanifold of a nearly Quasi Sasakian manifold with quarter-symmetric metric connection.

KEYWORDS: *CR*-submanifold, nearly Quasi Sasakian manifold, parallel distribution, ξ-vertical *CR*-submanifold and quarter-symmetric metric connection.

1. INTRODUCTION

In 1978, Bejancu introduced the nation of CR-submanifolds of a Kaehler manifolds [1]. Since then a number of authors extensively studied these submanifolds ([3],[4],[9],[10]). On the other hand, quasi Sasakian manifolds have been studied by Blair [8], Kanemaki [11], Tanno [19] and Rahman et. al [12-18]. Calin extensively studied integrability and geodesic property of the distribution of contact CR-submanifolds of quasi-Sasakian manifolds ([5],[6],[7]).

The purpose of the present paper is to study the notion of CR-submanifolds of a nearly quasi Sasakian manifold with the quarter-symmetric metric connection. The rest of this paper is organised as follows. In section 2 we recall some results and formula for the latter use. In section 3 we prove some basic lemma about CR-submanifold of a nearly quasi Sasakian manifold with the quarter-symmetric metric connection. In section 4 we study parallel distributing relating to ξ-verticle CR-submanifold of a nearly quasi Sasakian manifold with the quarter-symmetric metric connection. In section 5 we obtain Integrability conditions of distribution on CR-submanifold of a nearly quasi Sasakian manifold with the quarter-symmetric metric connection.

2. PRELIMINARIES

Let $\bar{M}$ be a real 2n+1 dimensional differentiable manifold, endowed with an almost contact metric structure (ϕ,ξ,η,g). then we have from [9]

$$\phi^2 = -I + \eta\otimes\xi,\ \eta(X) = g(X,\xi) \tag{1}$$

$$\eta(\xi) = 1, \qquad \eta o\phi = 0, \qquad \phi(\xi) = 0 \tag{2}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3}$$

For any vector field X, Y tanjent to $\bar{M}$, where I is the identity on the tanjent bundle $\Gamma\bar{M}$ of $\bar{M}$. In this case

$$g(\phi X,Y) = -g(X,\phi Y) \tag{4}$$

An almost contact metric structure (ϕ,ξ,η,g) on $\bar{M}$ is called quasi –sasakian manifold if

$$(\bar{\nabla}_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi,\ \phi AX = A\phi X \tag{5}$$

where A a symmetric linear transformation field.$\bar{\nabla}$ denotes the Riemannian connection of g on $\bar{M}$. On a quasi-sasakian manifold $\bar{M}$. We have

$$\bar{\nabla}_X\xi = \phi AX \tag{6}$$

Further, an almost contact metric manifold $\bar{M}$ on (ϕ,ξ,η,g) is called nearly quasi Sasakian manifold if

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi \tag{7}$$

Now, a quarter symmetric metric connection $\bar{\bar{\nabla}}$ on M is defined by

$$\bar{\bar{\nabla}}_X Y = \nabla_{\mathrm{X}} Y + \eta(Y)\phi X - g(\phi X,Y)\xi \tag{8}$$

such that $\bar{\bar{\nabla}}_X g = 0$. Using (1), (2) and (6) in (5) and (6), we get respectively

$$(\bar{\bar{\nabla}}_X\phi)X = \eta(Y)AX - g(AX,Y)\xi - g(X,Y)\xi + \eta(Y)X \tag{9}$$

In particular, an almost contact metric manifold $\bar{M}$ on (ϕ,ξ,η,g) is called nearly quasi-Sasakian manifold with quarter symmetric metric connection if

$$(\bar{\bar{\nabla}}_X\phi)Y + (\bar{\bar{\nabla}}_Y\phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi \\ -2g(X,Y)\xi + \eta(X)Y + \eta(Y)X \tag{10}$$

Now let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbole g. Let TM and $T^\perp M$ be the Lie algebras of vectors fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingraten formulas are given by

$$\bar{\bar{\nabla}}_X Y = \nabla_X Y + \hslash(X,Y) \tag{11}$$

$$\bar{\bar{\nabla}}_X N = -A_N X + \nabla^\perp_{\mathrm{X}} N + \eta(N)\phi X \tag{12}$$

For any $X, Y \epsilon TM$ and $N \epsilon\, T^\perp M$, where $\nabla^\perp$ is the connection on the normal bundle $T^\perp M$, $\hslash$ is the second fundamental from and A_N is the Weingraten map associated with N as

$$g(A_N X,Y) = g(\hslash(X,Y),N) \tag{13}$$

For any $x\epsilon M$ and $X\epsilon T_X M$, we write

$$X = PX + QX \tag{14}$$

---

[1] *Corresponding Author: Shamsur Rahman*
E-mail: shamsur@rediffmail.com

where $PX\epsilon D$ and $QX\epsilon D^{\perp}$. similarly for N normal to M, we have

$$\phi N = BN + CN \tag{15}$$

where BN (respectively CN) is the tangential component (respectively normal component) of ϕN.

Definition 2.1 An m dimensional Riemannian submanifold M of $\bar{M}$ is called a CR-submanifold of M if there exists a differentiable distribution $D: x \to D_X$ on M satisfying the following conditions:

(i) D is invariant , that is $\phi D_X \subset D_X$ for each $\epsilon M,$

(ii) The complementary orthogonal distribution $D^{\perp}: X \to D_X^{\perp} \subset T_X M\ of\ D$ is anti-invariant, that is, $\phi D_X^{\perp} \subset T_X^{\perp} M$ for each $x\epsilon M$. If $dim D_X^{\perp} = 0$, (respectly, $dim D_X = 0$) then the CR-submanifold is called an invariant (respectly, anti-invariant) submanifold. The distribution D (respectly, $D^{\perp}$) is called the horizontal (respectly, vertical) distribution. Also, the pair $(D, D^{\perp})$ is called ξ-horizontal (respectly, vertical) if $\xi_X \epsilon D_X$ (respectly, $\xi_X \epsilon D_X^{\perp}$).

3. MAIN RESULTS

Lemma 3.1 If M be a CR-submanifold of a nearly quasi Sasakian manifold with quarter symmetric metric connection $\bar{M}$, then

$$\begin{aligned}
&P(\bar{\nabla}_X \phi PY) + P(\bar{\nabla}_Y \phi PX) - P(A_{\phi QY} X) - P(A_{\phi QX} Y)\\
&\quad = -2g(PAX, Y)P\xi + \eta(Y)PAX + \eta(X)PAY\\
&\quad +\eta(X)PY + \eta(Y)PX + \phi P\nabla_X Y\\
&\quad +\phi P\nabla_Y X - 2g(PX, PY)P\xi \qquad (16)\\
&Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q\left(A_{\phi QY} X\right) - Q(A_{\phi QX} Y)\\
&\quad = 2Bh(X, Y) - 2g(QAX, Y)Q\xi + \eta(Y)QAX\\
&\quad +\eta(X)QAY + \eta(X)QY + \eta(Y)QX\\
&\quad -2g(QX, QY)Q\xi \qquad (17)\\
&h(X, \phi PY) + \hbar(Y, \phi PX) + \nabla_X^{\perp} \phi QY + \nabla_Y^{\perp} \phi QX\\
&\quad = \phi Q\nabla_Y X + \phi Q\nabla_X Y + 2C\hbar(X, Y)
\end{aligned}$$

for any $X, Y\epsilon TM$.

Proof: Using (8), (9), and (11) we get

$$\begin{aligned}
&(\bar{\nabla}_X \phi)Y + \phi(\nabla_X Y)Y + \phi h(X, Y)\\
&\quad = P\nabla_X(\phi PY) + Q\nabla_X(\phi PY) - PA_{\phi QY} X\\
&\quad -QA_{\phi QY} X + \nabla_X^{\perp}(\phi QY) + \hbar(X, \phi PY)
\end{aligned}$$

Interchange X and Y in the above equation and adding each other, using (5) and (12), we get

$$\begin{aligned}
&P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY} X - PA_{\phi QX} Y\\
&\quad +Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - QA_{\phi QY} X\\
&\quad -QA_{\phi QX} Y + h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^{\perp} \phi QY\\
&\quad +\nabla_Y^{\perp} \phi QX = 2Bh(X, Y) + 2Ch(X, Y)\\
&\quad +\eta(Y)PAX + \eta(Y)QAX + \eta(X)PAY\\
&\quad +\eta(X)QPY - 2g(PAX, Y)P\xi - 2g(QAX, Y)Q\xi\\
&\quad -2g(PX, PY)P\xi
\end{aligned} \tag{19}$$

For any $X, Y\epsilon TM$. Now equating horizontal, vertical, and normal components in (19), we get the desired result.

Lemma 3.2 If M be a CR-submanifold of a nearly quasi Sasakian manifold $\bar{M}$ with quarter symmetric metric connection, then

$$\begin{aligned}
&2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + \hbar(X, \phi Y)\\
&\quad -\hbar(Y, \phi X) - \phi[X, Y] + \eta(Y)AX + \eta(X)AY\\
&\quad -2g(AX, Y)\xi - \eta(Y)X + \eta(X)Y \qquad (20)\\
&2(\bar{\nabla}_Y \phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi\\
&\quad -2g(X, Y)\xi + \eta(Y)X + \eta(X)Y - \nabla_X \phi Y\\
&\quad +\nabla_Y \phi X - \hbar(X, \phi Y) + \hbar(Y, \phi X) + \phi[X, Y] \qquad (21)
\end{aligned}$$

Proof: From gauss formula (8), we have

$$\begin{aligned}
&\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = \bar{\nabla}_X \phi Y\\
&\quad +\hbar(X, \phi Y) - \nabla_Y \phi X - \hbar(Y, \phi X)
\end{aligned} \tag{22}$$

Also we have

$$\begin{aligned}
&\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y\\
&\quad -(\bar{\nabla}_Y \phi)X + \phi[X, Y]
\end{aligned} \tag{23}$$

From (22) and (23), we get

$$\begin{aligned}
&(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = \nabla_X \phi Y + \hbar(X, \phi Y)\\
&\quad -\nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]
\end{aligned} \tag{24}$$

Also for nearly quasi Sasakian manifold with quarter symmetric metric connection, we have

$$\begin{aligned}
&(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \eta(Y)AX + \eta(X)AY\\
&\quad -2g(X, Y)\xi + \eta(Y)X + \eta(X)Y
\end{aligned} \tag{25}$$

Adding (24) and (25), we get

$$\begin{aligned}
&2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + \hbar(X, \phi Y)\\
&\quad -\hbar(Y, \phi X) - \phi[X, Y] + \eta(Y)AX + \eta(X)AY\\
&\quad -2g(AX, Y)\xi - \eta(Y)X + \eta(X)Y
\end{aligned} \tag{20}$$

Subtracting (24) and (25), we get

$$\begin{aligned}
&2(\bar{\nabla}_Y \phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi\\
&\quad -2g(X, Y)\xi + \eta(Y)X + \eta(X)Y - \nabla_X \phi Y\\
&\quad +\nabla_Y \phi X - \hbar(X, \phi Y) + \hbar(Y, \phi X) + \phi[X, Y]
\end{aligned} \tag{21}$$

Lemma 3.3: If M be a CR-submanifold of a nearly quasi Sasakian manifold $\bar{M}$, with quarter symmetric metric connection, then

$$\begin{aligned}
&2(\bar{\nabla}_Y \phi)Z = A_{\phi Y} Z - A_{\phi Z} Y - \nabla_Z^{\perp} \phi Y + \nabla_Y^{\perp} \phi Z\\
&\quad -\phi[Y, Z] + \eta(Z)AY + \eta(Y)AZ - 2g(AY, Z)\xi\\
&\quad -2g(Y, Z)\xi + \eta(Z)Y + \eta(Y)Z\\
&2(\bar{\nabla}_Z \phi)Y = \eta(Z)Y + \eta(Y)Z - 2g(AY, Z)\xi\\
&\quad -2g(Y, Z)\xi + \eta(Z)AY + \eta(Y)AZ\\
&\quad + A_{\phi Z} Y - A_{\phi Y} Z + \nabla_Z^{\perp} \phi Y - \nabla_Y^{\perp} \phi Z + \phi[Y, Z]
\end{aligned}$$

for any $Y, Z \epsilon D^{\perp}$.

Proof. From Weingarten formula (13), we have

$$\bar{\nabla}_Z \phi Y - \bar{\nabla}_Y \phi Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla_Y^{\perp} \phi Z - \nabla_Z^{\perp} \phi Y \tag{26}$$

Also, we have

$$\bar{\nabla}_Z \phi Y - \bar{\nabla}_Y \phi Z = (\bar{\nabla}_Y \phi)Z - (\bar{\nabla}_Z \phi)Y + \phi[Y, Z] \tag{27}$$

From (26) and (27), we get

$$\begin{aligned}
&(\bar{\nabla}_Y \phi)Z - (\bar{\nabla}_Z \phi)Y = A_{\phi Y} Z - A_{\phi Z} Y\\
&\quad +\nabla_Y^{\perp} \phi Z - \nabla_Z^{\perp} \phi - \phi[Y, Z]
\end{aligned} \tag{28}$$

Also for nearly quasi Sasakian manifold $\bar{M}$, with quarter symmetric metric connection then, we have

$$\begin{aligned}
&(\bar{\nabla}_Y \phi)Z + (\bar{\nabla}_Z \phi)Y = \eta(Z)AY + \eta(Y)AZ - 2g(AY, Z)\xi\\
&\quad -2g(Y, Z)\xi + \eta(Z)Y + \eta(Y)Z
\end{aligned} \tag{29}$$

Adding (28) and (29), we get

$$\begin{aligned}
&2(\bar{\nabla}_Y \phi)Z = A_{\phi Y} Z - A_{\phi Z} Y - \nabla_Z^{\perp} \phi Y + \nabla_Y^{\perp} \phi Z\\
&\quad -\phi[Y, Z] + \eta(Z)AY + \eta(Y)AZ - 2g(AY, Z)\xi\\
&\quad -2g(Y, Z)\xi + \eta(Z)Y + \eta(Y)Z
\end{aligned}$$

Subtracting (3.13) from (3.14) we get

$$\begin{aligned}
&2(\bar{\nabla}_Z \phi)Y = \eta(Z)Y + \eta(Y)Z - 2g(AY, Z)\xi\\
&\quad -2g(Y, Z)\xi + \eta(Z)AY + \eta(Y)AZ\\
&\quad + A_{\phi Z} Y - A_{\phi Y} Z + \nabla_Z^{\perp} \phi Y - \nabla_Y^{\perp} \phi Z + \phi[Y, Z]
\end{aligned}$$

This proves our assertions.

Lemma 3.4 If M be a CR-submanifold of a nearly quasi Sasakian manifold $\bar{M}$, with quarter symmetric metric connection, then

$$\begin{aligned}
&2(\bar{\nabla}_X \phi)Y = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi\\
&\quad -2g(X, Y)\xi + \eta(Y)X + \eta(X)Y - A_{\phi Y} X + \nabla_X^{\perp} \phi Y\\
&\quad -\nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]\\
&2(\bar{\nabla}_Y \phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi\\
&\quad -2g(X, Y)\xi + \eta(Y)X + \eta(X)Y + A_{\phi Y} X - \nabla_X^{\perp} \phi Y\\
&\quad +\nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y]
\end{aligned}$$

for any $X \in D$and $Y \in D^{\perp}$.

Proof. By using Gauss equation and Weingarten equation for $X \in D$and $Y \in D^{\perp}$ respectively we get

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) \tag{30}$$

Also, we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[X,Y] \tag{31}$$

From (30) and (31), we get

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] \tag{32}$$

Also for nearly quasi Sasakian manifold $\overline{M}$, with quarter symmetric metric connection, we have

$$(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi - 2g(X,Y)\xi + \eta(X)Y + \eta(Y)X \tag{33}$$

Adding (32) and (33), we get

$$2(\overline{\nabla}_X\phi)Y = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi - 2g(X,Y)\xi + \eta(Y)X + \eta(X)Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

Subtracting (32) from (33) we get

$$2(\overline{\nabla}_Y\phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi - 2g(X,Y)\xi + \eta(Y)X + \eta(X)Y + A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y]$$

Hence Lemma is proved.

4. PARALLEL DISTRIBUTIONS

Definition 4.1. The horizontal (respectly, vertical) distribution D (respectly, $D^{\perp}$) is said to be parallel [1] with respect to the connection on M if $\nabla_X Y \in D$ (respectly, $\nabla_Z W \in D^{\perp}$) for any vector field $X, Y \in D$ (respectly, $W, Z \in D^{\perp}$).

Proposition 4.2. If M be a ξ-vertical CR-submanifold of a nearly quasi Sasakian manifold $\overline{M}$, with quarter symmetric metric connection and the horizontal distribution D is parallel, then

$$h(X,\phi Y) = h(Y,\phi X) \tag{34}$$

for all $X,\ Y \in D$.

Proof. Using parallelism of horizontal distribution D, we have

$$\nabla_X\phi Y \in D,\ \nabla_Y\phi X \in D \quad \text{for any } X,\ Y \in D. \tag{35}$$

Thus using the fact that $X = QY = 0$ for $Y \in D$, (17) gives

$$Bh(X,Y) = g(AX,Y)Q\xi \quad \text{for any } X,\ Y \in D. \tag{36}$$

Also, since

$$\phi h(X,Y) = Bh(X,Y) + Ch(X,Y), \tag{37}$$

then

$$\phi h(X,Y) = g(AX,Y)Q\xi + Ch(X,Y) \text{ for any } X,\ Y \in D. \tag{38}$$

Next from (18), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2Ch(X,Y) = 2\phi h(X,Y) - 2g(AX,Y)Q\xi \tag{39}$$

for any $X,\ Y \in D$. Putting $X = \phi X \in D$ in (39), we get

$$h(\phi X,\phi Y) + h(Y,\phi^2 X) = 2\phi h(\phi X,Y) - 2g(A\phi X,Y)Q\xi \tag{40}$$

or

$$h(\phi X,\phi Y) - h(Y,X) = 2\phi h(\phi X,Y) - 2g(A\phi X,Y)Q\xi \tag{39}$$

Similarly, putting $Y = \phi Y \in D$ in (37), we get

$$h(\phi Y,\phi X) - h(X,Y) = 2\phi h(X,\phi Y) - 2g(AX,\phi Y)Q\xi. \tag{40}$$

Hence from (39) and (40), we have

$$\phi h(X,\phi Y) - \phi h(Y,\phi X) = g(AX,\phi Y)Q\xi - g(A\phi X,Y)Q\xi \tag{41}$$

Operating ϕ on both sides of (41) and using $\phi\xi = 0$, we get

$$h(X,\phi Y) = h(Y,\phi X) \tag{42}$$

for all $X,\ Y \in D$.

Now, for the distribution $D^{\perp}$, we prove the following proposition.

Proposition 4.3. Let M be a ξ-vertical CR-submanifold of a nearly quasi Sasakian manifold $\overline{M}$, with quarter symmetric metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M, then

$$\mathrm{A}_{\phi Y}Z + \mathrm{A}_{\phi Z}Y \in D^{\perp} \quad \text{for any } Y, Z \in D^{\perp}. \tag{43}$$

Proof. Let $Y, Z \in D^{\perp}$, then using Gauss and Weingarten formula (10),
we obtain

$$-\mathrm{A}_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \mathrm{A}_{\phi Y}Z + \nabla_Z^{\perp}\phi Y = \phi\nabla_Y Z + \phi h(Y,Z) + \phi\nabla_Z Y + \phi h(Z,Y) + 2g(AY,Z)\xi + \eta(Y)AZ + \eta(Z)AY \tag{44}$$

for any $Y, Z \in D^{\perp}$. Taking inner product with $X \in D$ in (44), we get

$$g(\mathrm{A}_{\phi Y}Z, X) + g(\mathrm{A}_{\phi Z}Y, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X) \tag{45}$$

If the distribution $D^{\perp}$ is parallel, then $\nabla_Y Z \in D^{\perp}$ and $\nabla_Z Y \in D^{\perp}$, for any $Y, Z \in D^{\perp}$.
So from (45) we get

$$g(\mathrm{A}_{\phi Y}Z, X) + g(\mathrm{A}_{\phi Z}Y, X) = 0$$

$$\text{or} \quad g(\mathrm{A}_{\phi Y}Z + \mathrm{A}_{\phi Z}Y, X) = 0 \tag{46}$$

which is equivalent to

$$\mathrm{A}_{\phi Y}Z + \mathrm{A}_{\phi Z}Y \in D^{\perp} \text{ for any } Y, Z \in D^{\perp} \tag{47}$$

and this completes the proof.

Definition 4.4. A CR-submanifold M is said to be mixed totally geodesic if $h(X,Z) = 0$ for $X \in D$ and $Z \in D^{\perp}$.
The following lemma is an easy consequences of (10).

Lemma 4.5. If M be a CR-submanifold of a nearly quasi Sasakian manifold $\overline{M}$, with quarter symmetric metric connection, then M is mixed if and only if $A_N X \in D$ for all $X \in D$.

Definition 4.6. A normal vector field $N \neq 0$is called D-parallel normal section if $\nabla_X^{\perp}N = 0$ for all $X \in D$.

Proposition 4.7 Let M be a mixed totally geodesic ξ-vertical CR-submanifold M of a nearly quasi Sasakian manifold $\overline{M}$, with quarter symmetric metric connection. Then from normal section $N \in \phi D^{\perp}$ is D parallel if and only if $\nabla_X\phi N \in D$
for all $X \in D$.

Proof: Let $N \in \phi D^{\perp}$ then from (17), we have

$$Q(\nabla_Y\phi X) = 0 \quad \text{for any } X \in D, Y \in D^{\perp} \tag{48}$$

In particular, we have $Q(\nabla_Y X) = 0$. By using it in (18), we get

$$\nabla_X^{\perp}\phi QY = \phi Q\nabla_Y X \ \text{ or } \nabla_X^{\perp}N = -\phi Q\nabla_X\phi N \tag{49}$$

Thus if the normal section $N \neq 0$ is D-parallel, then uding Definition 4 and (49), we get

$$\phi Q(\nabla_X\phi N) = 0 \tag{50}$$

Which is equivelent to $\nabla_X\phi N \in D$ for all $X \in D$. The converse part easily follows from (49).
This complets the proof of the proposition.

5. Integrability conditions of distributions

Lemma 5.1. If M be a CR-submanifold of a nearly quasi Sasakian manifold $\overline{M}$, with quarter symmetric metric connection, then

$$(\overline{\nabla}_{\phi X}\phi)Y = \eta(Y)A\phi X - 2g(A\phi X,Y)\xi - 2g(\phi X,Y)\xi + \eta(Y)\phi X - \eta(X)\overline{\nabla}_Y\xi + \phi(\overline{\nabla}_Y\phi)(X) + \eta(\overline{\nabla}_Y X)\xi \tag{51}$$

for any $X, Y \in TM$.

Proof. For nearly quasi Sasakian manifold $\overline{M}$, with quarter symmetric metric connection, we have

$$\begin{aligned}(\bar{\nabla}_{\phi X}\phi)Y &= \eta(Y)A\phi X - 2g(A\phi X,Y)\xi \\ &\quad -2g(\phi X,Y)\xi + \eta(Y)\phi X - (\bar{\nabla}_{\phi Y}\phi)\phi Y \qquad (52)\end{aligned}$$

and we have

$$\begin{aligned}(\bar{\nabla}_{Y}\phi)\phi X &= \bar{\nabla}_{Y}\phi^{2}X - \phi(\bar{\nabla}_{Y}\phi X) \\ &= \bar{\nabla}_{Y}\phi^{2}X - \phi(\bar{\nabla}_{Y}\phi X) + \phi(\phi\bar{\nabla}_{Y}X) - \phi(\phi\bar{\nabla}_{Y}X) \\ &= -\bar{\nabla}_{Y}X + \eta(X)\bar{\nabla}_{Y}\xi - \phi(\bar{\nabla}_{Y}\phi X \\ &\quad -\phi\bar{\nabla}_{Y}X) - \phi(\phi\bar{\nabla}_{Y}X)\end{aligned}$$

$$(\bar{\nabla}_{Y}\phi)\phi X = \eta(X)\bar{\nabla}_{Y}\xi - \phi(\bar{\nabla}_{Y}\phi)(X) - \eta(\phi\bar{\nabla}_{Y}X)\xi \qquad (53)$$

by (53) in (52), we have

$$\begin{aligned}(\bar{\nabla}_{\phi X}\phi)Y &= \eta(Y)A\phi X - 2g(A\phi X,Y)\xi \\ &\quad -2g(\phi X,Y)\xi + \eta(Y)\phi X - \eta(X)\bar{\nabla}_{Y}\xi \\ &\quad +\phi(\bar{\nabla}_{Y}\phi)(X) + \eta(\bar{\nabla}_{Y}X)\xi \qquad (54)\end{aligned}$$

for any $X, Y \epsilon TM$, which completes the proof of the lemma. On a nearly quasi Sasakian manifold $\bar{M}$, with quarter symmetric metric connection, Nijenhuis tensor is given by

$$\begin{aligned}N_{\phi}(X,Y) &= (\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X \\ &\quad -\phi(\bar{\nabla}_{X}\phi)Y + \phi(\bar{\nabla}_{Y}\phi)X \qquad (55)\end{aligned}$$

for any $X, Y \epsilon TM$.

As of(51) and (55), we have

$$\begin{aligned}N_{\phi}(X,Y) &= -\eta(Y)A\phi X - \eta(X)A\phi Y - 4g(A\phi X,Y)\xi \\ &\quad -4g(\phi X,Y)\xi - \eta(Y)\phi X - 3\eta(X)\phi Y - \eta(X)\bar{\nabla}_{Y}\xi \\ &\quad +\eta(Y)\bar{\nabla}_{X}\xi + \eta(\bar{\nabla}_{Y}X)\xi - \eta(\bar{\nabla}_{X}Y)\xi + 4\phi(\bar{\nabla}_{Y}\phi)X\end{aligned}$$

Proposition 5.2. If M be a nearly quasi Sasakian manifold $\bar{M}$, with quarter symmetric metric connection, then

$$\begin{aligned}3(A_{\phi Y}Z - A_{\phi Z}Y) &= \phi P[Y,Z] \\ &\quad +3[\eta(Z)(AY+Y) - \eta(Y)(AZ+Z)] \qquad (56)\end{aligned}$$

for any $Y, Z \epsilon D^{\perp}$.

Proof: For $Y, Z \epsilon D^{\perp}$ and $X \epsilon T(M)$, we have

$$\begin{aligned}2g(A_{\phi Z}Y,\ X) &= 2g(h(X,Y),\phi Z) \\ &= g(h(X,Y),\phi Z) + g(h(X,Y),\phi Z) \qquad (57) \\ &= -g(\bar{\nabla}_{X}\phi Y,Z) - g(\bar{\nabla}_{Y}\phi X,Z) + g(\phi(\bar{\nabla}_{X}Y + \bar{\nabla}_{Y}X),Z)\end{aligned}$$

The above equation is true for all $X \epsilon T(M)$, therefore transvecting the vector field X both sides, we have

$$\begin{aligned}2A_{\phi Z}Y &= A_{\phi Y}Z - \phi\bar{\nabla}_{Y}Z + \eta(Y)AZ \\ &\quad +g(AY,Z)\xi - 2\eta(Z)AY - 2\eta(Z)Y \\ &\quad +\eta(Y)Z + g(Y,Z)\xi \qquad (58)\end{aligned}$$

for any $Y, Z \epsilon D^{\perp}$. Interchanging the vector fields Y and Z, we get

$$\begin{aligned}2A_{\phi Y}Z &= A_{\phi Z}Y - \phi\bar{\nabla}_{Z}Y + \eta(Z)AY \\ &\quad +g(AZ,Y)\xi - 2\eta(Y)AZ - 2\eta(Y)Z \\ &\quad +\eta(Z)Y + g(Z,Y)\xi \qquad (59)\end{aligned}$$

Subtracting (58) and (59), we get

$$\begin{aligned}3(A_{\phi Y}Z - A_{\phi Z}Y) &= \phi P[Y,Z] \\ &\quad +3[\eta(Z)(AY+Y) - \eta(Y)(AZ+Z)] \qquad (60)\end{aligned}$$

for any $Y, Z \epsilon D^{\perp}$.

Theorem 5.3. If Mbe a CR-submanifold of a nearly quasi Sasakian manifold $\bar{M}$, with quarter symmetric metric connection, then the distribution $D^{\perp}$ is integrable if and only if

$$(A_{\phi Y}Z - A_{\phi Z}Y) = [\eta(Z)(AY+Y) - \eta(Y)(AZ+Z)] \qquad (61)$$

Proof: Primary suppose that the distribution $D^{\perp}$ is integrable. Then $[Y,Z] \epsilon D$ for any $Y, Z \epsilon D^{\perp}$. Since P is a projection operator on D, so $P[Y,Z] = 0$. Thus from (56) we get (61). Conversely, we suppose that (61) holds. Then using (56), we have $\phi P[Y,Z] = 0$ for any $Y, Z \epsilon D^{\perp}$. Since $rank\ \phi = 2n$. Therefore, either $P[Y,Z] = 0$ or $P[Y,Z] = k\xi$. But $P[Y,Z] = k\xi$ is not possible as P is a projection operator on D. Thus, $P[Y,Z] = 0$, which is equivalent to $[Y,Z] \epsilon D^{\perp}$ for any $Y, Z \epsilon D^{\perp}$ and hence $D^{\perp}$ is integrable.

Corollary 5.4. If Mbe a ξ-horizontal CR-submanifold of a nearly quasi Sasakian manifold $\bar{M}$, with quarter symmetric metric connection, then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \qquad (62)$$

REFERENCES

[1] Bejancu, A. CR-submanifold of a Kaehler manifold, I, Proe. Amer. Math. Soc. Vol. 69 (1), pp.135-142 (1978).

[2] Bejancu, A. and Papaghuie, N., CR-submanifold of Kenmotsu manifold, Rend. Mat Vol. 7, pp. 607-622 (1984).

[3] Chen, B. Y., CR-submanifold of a Kaehlarian manifold I, J. Differential Geometry, Vol.16 pp.305-322 (1981).

[4] Chen, B. Y., CR-submanifold of a Kaehlarian manifold II, J. Differential geometry, Vol. 16, pp. 493-509 (1981).

[5] Calin, C., Contact CR-submanifold of quasi-Sasakian manifolds, Bull. Math. Soc. Sci. Math. Roumanic Vol 36(84), pp. 217-226 (1992).

[6] Calin, C. Geometry of leaves on a CR-submanifold of a quasi Sasakian manifold, Bul. I.P. Iasi Tom XL(XLIV) Fasc. Vol 1-4, pp. 257-270 (1994).

[7] Calin, C., Normal contact CR-submanifolds of a quasi-Sasakian manifolds, Publ. Math. Debreeen Vol. 53/3-4, pp.257-270 (1998).

[8] Blair, D. E., The theory of quasi-Sasakian structures, J. Differential Geometry, Vol. I, pp. 331-345 (1967).

[9] Blair, D. E., Contact Manifolds in Riemannian Geometry, Lectures Notes in Mathematics, vol. 509, Springer- Verlag, Berlin, 1976.

[10] Blair, D. E. and Chen, B. Y. On CR-submanifolds of Hermitian manifolds, Israel J. Math Vol. 34, 353-363 (1979).

[11] Blair, D. E., Showers, D. K. and Yano, K., Nearly Sasakian structure, Kodai Math. Sem. Rep. Vol 27, pp. 175-180, (1976).

[12] Kanemaki, S., Quasi Sasakian manifolds, Tohoku Math. Journ. Vol -29, pp. 227-233 (1977).

[13] Rahman S and Ahmad, A., On the Geometry of Hypersurfaces of a Certain connection in a Quasi-sasakian Manifolds. International Journal Mathematical Combinatorics, Vol. 3, pp. 23-33 (2011).

[14] Rahman, S., and Shafiullah, Geometry of Hypersurfaces of a Semi Symmetric Semi Metric Connection in a Quasi-Sasakian Manifold. Journal of Purvanchal Academy of Sciences, Vol. 17, pp. 231-242, (2011).

[15] Rahman, S. and Ahmad, A., On The Geometry of Hypersurfaces of a Certain Connection in a Quasi-Sasakian Manifold, International Journal Mathematical Combinatorics Vol.3, pp. 23-33 (2011)**.**

[16] Rahman, S., Geometry of Hypersurfaces of a semi symmetric metric connection in a quasi-Sasakian manifold, Journal-Proceedings of the Institute of Applied Mathematics, Vol.3 (2), pp.152-164, (2014).

[17] Rahman, S. Geometry of hypersurfaces of a quarter symmetric non metric connection in a quasi-Sasakian manifold. Carpathian Mathematical Publications Vol. 7(2) pp. 226-235 (2015).

[18] Rahman, S., Contact conformal connection on a geometry of hypersurfaces with certain connection in a quasi-Sasakian manifold, Bulletin of the Transilvania University of Brasov Series III: Mathematics, Informatics, Physics, Vol 10 (59), No. 1, pp.135-148, (2017).

[19] Tanno, S., Quasi-Sasakian structures of rank 2p+1, J. Differential Geometry, Vol. 5, pp. 317-324 (1971).