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(*p*,*q*)- GENERALIZATION OF A CLASS OF MODIFIED BERNSTEIN-TYPE OPERATORS

M. A. Siddiqui[1],Nandita Gupta[2]and Maheshwari Sahu[3]
[1 &2]Department Of Mathematics,Govt. V. Y. T. P. G. Autonomous College, Durg, Chhattisgarh, India
[3]Department of Applied Mathematics, RSR Rungta College, Durg, Chhattisgarh, India
E-mail: [1]dr_m_a_siddiqui@yahoo.co.in ,[2]nandita_dec@yahoo.com , [3]mahisahu86@gmail.com

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ABSTRACT: In the present paper we propose a (*p*,*q*)-variant of modified Bernstein type operators and give the convergence theorem for these operators. We also investigate rate of convergence of these new operators using modulus of continuity.

KEYWORDS: Linear positive operator, Bernstein operators, *q* - integers, (*p*,*q*)- integers.

1. INTRODUCTION

In the year 2008, Deo et al. [4] introduced the class of new Bernstein type operators as

$$(V_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)$$

where $f \in C\left[0, \frac{n}{n+1}\right], n \in \mathbb{N}$and

$$p_{n.k}(x) = \left(\frac{1+n}{n}\right)^n x^k \binom{n}{k} \left(\frac{n}{n+1} - x\right)^{n-k}, x \in \left[0, \frac{n}{n+1}\right] \quad (1.1)$$

Many modifications of these operators have been introduced andstudied. In 2014, Siddiqui et al. [16]introduced aclass of modified new Bernstein type operators as

$$V_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n+1}\right), x \in \left[0, \frac{n}{n+1}\right] \quad (1.2)$$

where $f \in C\left[0, \frac{n}{n+1}\right]andp_{n,k}(x)$ aresame as in equation (1.1). Note that these operators are defined onthe mobile interval$\left[0, \frac{n}{n+1}\right], n \in \mathbb{N}$ and for sufficiently large n, they convert in the classical Bernstein operators.

A q-analogue of operators (1.2) were introduced as

$$V_n^*(f;x) = \left(\frac{n+1}{n}\right)^n \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n+1]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} \left(\frac{n}{n+1} - q^s x\right) \quad (1.3)$$

where $x \in \left[0, \frac{n}{n+1}\right]$.

In 2015, Mursaleen et al. [11] applied (*p*,*q*)-calculus in approximationtheory. They introduced (*p*,*q*)-analogues of Bernstein operators asfollows

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \times \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \quad (1.4)$$

where $x \in [0,1]$. Later on they introduced (*p*,*q*)-analogues of various operatorssuch as Bernstein-Stancu operators [12],Bernstein-Schurer operators [13],Kantorovichtype Bernstein operators [14], etc. Acar [1] introduced (*p*,*q*)-variant of Sz´asz-Mirakyan operators.

In this paper we propose a (*p*,*q*)-variant of the operators (1.2). For *p*=1 theseoperators will reduce to the q-analogue of the operators (1.2).We recall some basic notions based on (*p*,*q*)-integers.

Let $0 < q < p \leq 1$. For each nonnegative integer $k, n, n \geq k \geq 0$, the (*p*, *q*)-integer $[k]_{p,q}$, (*p*, *q*)-factorial$[k]_{p,q}!$ and (*p*, *q*)-binomial coefficient are defined by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}$$

with $[k]_{p,q} = 1$ for k = 0.

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q} \ldots 1, k \geq 1 \\ 1 \qquad\qquad\qquad\quad , k = 0 \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

The (*p*,*q*)-Binomial expansion is

$$(ax+by)_{p,q}^n = \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} p^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k$$

and

$$(x+y)_{p,q}^n = (x+y)(px+qy)(p^2x+q^2y) \ldots (p^{n-1}x+q^{n-1}y)$$

Corresponding Author:Nandita Gupta[2]
Email: nandita_dec@yahoo.com

For $p = 1$ the above notations reduce to their q-analogues. More details on (p, q)-calculus can be found in [2], [3], [5], [6], [7], [9],[15].

Let$0 < q < p \leq 1$ and $n \in \mathbb{N}$. For $f \in C\left[0, \frac{n}{n+1}\right]$wedefine

$$V_{n,p,q}^{*}(f;x) = \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^{k} \times \prod_{s=0}^{n-k-1} \left(\frac{np^{s}}{n+1} - q^{s}x\right) f\left(\frac{[k]_{p,q}}{p^{k-n}[n+1]_{p,q}}\right) \quad (2.1)$$

where $x \in \left[0, \frac{n}{n+1}\right]$ and an empty product denotes1. Clearly these are linear positive operators and for p=1 they reduce to the operators (1.3) defined in [16].For sufficiently large n these operators reduce to the operators (1.4).

Lemma 2.1. Let $0 < q < p \leq 1$, $f \in C\left[0, \frac{n}{n+1}\right]$and $n \in \mathbb{N}$.Then

$$V_{n,p,q}^{*}(f;0) = f(0) \text{ and } V_{n,p,q}^{*}\left(f;\frac{n}{n+1}\right) = f\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right)$$

Proof. By (2.1) we get

$$V_{n,p,q}^{*}(f;x) = \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} \prod_{s=0}^{n-1} \left(\frac{np^{s}}{n+1} - q^{s}x\right) f(0) + \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^{k} \times \prod_{s=0}^{n-k-1} \left(\frac{np^{s}}{n+1} - q^{s}x\right) f\left(\frac{[k]_{p,q}}{p^{k-n}[n+1]_{p,q}}\right)$$

On taking $x = 0$, we see that the terms after the first term vanish. So we get

$$V_{n,p,q}^{*}(f;0) = \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} \prod_{s=0}^{n-1} \left(\frac{np^{s}}{n+1} - q^{s}0\right) f(0) + 0 + 0 + \cdots$$

$$= \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \prod_{s=0}^{n-1} \left(\frac{np^{s}}{n+1}\right) f(0)$$

$$= f(0)$$

Again

$$V_{n,p,q}^{*}(f;x) = \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^{k} \times \prod_{s=0}^{n-k-1} \left(\frac{np^{s}}{n+1} - q^{s}x\right) f\left(\frac{[k]_{p,q}}{p^{k-n}[n+1]_{p,q}}\right) + \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ n \end{bmatrix}_{p,q} p^{\frac{n(n-1)}{2}} x^{n} f\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right) \times \prod_{s=0}^{-1} \left(\frac{np^{s}}{n+1} - q^{s}x\right)$$

On taking $x = \frac{n}{n+1}$ we obtain

2.CONSTRUCTION OF OPERATORS AND AUXILIARY RESULTS

$$V_{n,p,q}^{*}\left(f;\frac{n}{n+1}\right)$$

$$= \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{n}{n+1}\right)^{k} f\left(\frac{[k]_{p,q}}{p^{k-n}[n+1]_{p,q}}\right) \times \prod_{s=0}^{n-k-1} \left(\frac{np^{s}}{n+1} - \frac{nq^{s}}{n+1}\right) + \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ n \end{bmatrix}_{p,q} p^{\frac{n(n-1)}{2}} \times \left(\frac{n}{n+1}\right)^{n} f\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right) \prod_{s=0}^{-1} \left(\frac{np^{s}}{n+1} - \frac{nq^{s}}{n+1}\right)$$

In the summation on right side, the terms corresponding to$k = 0, 1, 2, \ldots, n-1$ become zero because for s=0, we find$\left(\frac{n}{n+1} - \frac{n}{n+1}\right)$ as first factor of eachproduct. So only the last term is left. Thus we have

$$V_{n,p,q}^{*}\left(f;\frac{n}{n+1}\right) = \frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ n \end{bmatrix}_{p,q} p^{\frac{n(n-1)}{2}} \left(\frac{n}{n+1}\right)^{n} \times f\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right) = f\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right) \quad \square$$

Lemma 2.2. Let$0 < q < p \leq 1$, $x \in \left[0, \frac{n}{n+1}\right]$and $n \in \mathbb{N}$.Then

$$\frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^{k} \prod_{s=0}^{n-k-1} \left(\frac{np^{s}}{n+1} - q^{s}x\right) = 1$$

Proof. We know that (see [11])

$$\frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} y^{k} \prod_{s=0}^{n-k-1} (p^{s} - q^{s}y) = 1 \quad (2.2)$$

where $y \in [0,1]$.

We put $y = \left(\frac{n+1}{n}\right)x$. Since $0 \leq x \leq \frac{n}{n+1}$, we find that $0 \leq y \leq 1$. Thus from (2.2) we obtain

$$\frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \left(\frac{n+1}{n}\right)^{k} x^{k} \times \prod_{s=0}^{n-k-1} \left(\frac{n+1}{n}\right)^{n-k} \left(\frac{np^{s}}{n+1} - q^{s}x\right) = 1$$

Thus

$$\frac{\left(\frac{n+1}{n}\right)^{n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^{k} \prod_{s=0}^{n-k-1} \left(\frac{np^{s}}{n+1} - q^{s}x\right) = 1$$

□

Lemma 2.3. Let $0 < q < p \leq 1$ and $n \in \mathbb{N}$ and $V_{n,p,q}^{*}(f)$ defined in (2.1) , then following equalities hold

$(i)\quad V_{n,p,q}^{*}(e_0;x)=1$

$(ii)\quad V_{n,p,q}^{*}(e_1;x)=\left(\frac{n+1}{n}\right)\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right)x$

$(iii)\quad V_{n,p,q}^{*}(e_2;x)=\left(\frac{n+1}{n}\right)^2\left(\frac{[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}\right)qx^2+\left(\frac{n+1}{n}\right)\left(\frac{p^{n-1}[n]_{p,q}}{[n+1]_{p,q}^2}\right)x$

where $e_i(t)=t^i$, $i=0,1,2$.

Proof. By virtue of Lemma 2.2 , from (2.1) we get

$$V_{n,p,q}^{*}(e_0;x)=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}x^k\times\prod_{s=0}^{n-k-1}\left(\frac{np^s}{n+1}-q^sx\right)=1$$

and

$$\begin{aligned}
&V_{n,p,q}^{*}(e_1;x)\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}\left(\frac{np^s}{n+1}-q^sx\right)\times\left(\frac{[k]_{p,q}}{p^{k-n}[n+1]_{p,q}}\right)\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n(n-1)}{2}}}\sum_{k=1}^{n}\frac{p^{\frac{k(k-1)}{2}}[n]_{p,q}!}{[k-1]_{p,q}![n-k]_{p,q}!}x^k\left(\frac{1}{p^{k-n}[n+1]_{p,q}}\right)\times\prod_{s=0}^{n-k-1}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n(n-1)}{2}}}\frac{[n]_{p,q}x}{[n+1]_{p,q}}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}\left(\frac{p^{\frac{k(k+1)}{2}}}{p^{k+1-n}}\right)x^k\times\prod_{s=0}^{n-k-2}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{(n-1)(n-2)}{2}}}\frac{[n]_{p,q}x}{[n+1]_{p,q}}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}x^k\times\prod_{s=0}^{n-k-2}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\left(\frac{n+1}{n}\right)\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right)x
\end{aligned}$$

Again from (2.1) using the fact that $[k+1]_{p,q}=p^k+q[k]_{p,q}$

$$\begin{aligned}
V_{n,p,q}^{*}(e_2;x)&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}x^k\times\prod_{s=0}^{n-k-1}\left(\frac{np^s}{n+1}-q^sx\right)\left(\frac{[k]_{p,q}}{p^{k-n}[n+1]_{p,q}}\right)^2\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n^2-5n}{2}}}\sum_{k=1}^{n}\frac{p^{\frac{k^2-5k}{2}}[n]_{p,q}!\,[k]_{p,q}x^k}{[k-1]_{p,q}![n-k]_{p,q}![n+1]_{p,q}^2}\times\prod_{s=0}^{n-k-1}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n^2-5n}{2}}}\sum_{k=0}^{n-1}\frac{p^{\frac{(k+1)(k-4)}{2}}[n]_{p,q}!\,[k+1]_{p,q}x^{k+1}}{[k]_{p,q}![n-k-1]_{p,q}![n+1]_{p,q}^2}\times\prod_{s=0}^{n-k-2}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n^2-5n}{2}}}\sum_{k=0}^{n-1}\frac{p^{\frac{(k+1)(k-4)}{2}}[n]_{p,q}!\,(p^k+q[k]_{p,q})x^{k+1}}{[k]_{p,q}![n-k-1]_{p,q}![n+1]_{p,q}^2}\times\prod_{s=0}^{n-k-2}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n^2-5n+4}{2}}}\frac{x[n]_{p,q}}{[n+1]_{p,q}^2}\sum_{k=0}^{n-1}p^{\frac{k(k-1)}{2}}x^k\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}\times\prod_{s=0}^{n-k-2}\left(\frac{np^s}{n+1}-q^sx\right)+\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{n^2-5n+6}{2}}}\\
&\quad\times\frac{[n]_{p,q}[n-1]_{p,q}qx^2}{[n+1]_{p,q}^2}\sum_{k=0}^{n-2}\begin{bmatrix}n-2\\k\end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}\times x^k\prod_{s=0}^{n-k-3}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\frac{p^{n-1}x[n]_{p,q}}{[n+1]_{p,q}^2}\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{(n-1)(n-2)}{2}}}\sum_{k=0}^{n-1}p^{\frac{k(k-1)}{2}}x^k\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}\times\prod_{s=0}^{n-k-2}\left(\frac{np^s}{n+1}-q^sx\right)+\frac{[n]_{p,q}[n-1]_{p,q}qx^2}{[n+1]_{p,q}^2}\\
&\quad\times\frac{\left(\frac{n+1}{n}\right)^n}{p^{\frac{(n-2)(n-3)}{2}}}\sum_{k=0}^{n-2}\begin{bmatrix}n-2\\k\end{bmatrix}_{p,q}p^{\frac{k(k-1)}{2}}x^k\times\prod_{s=0}^{n-k-3}\left(\frac{np^s}{n+1}-q^sx\right)\\
&=\left(\frac{n+1}{n}\right)^2\left(\frac{[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}\right)qx^2+\left(\frac{n+1}{n}\right)\left(\frac{p^{n-1}[n]_{p,q}}{[n+1]_{p,q}^2}\right)x
\end{aligned}$$

□

Lemma 2.4. Let $V^*_{n,p,q}(.;.)$ be given by (2.1), then for any $x \in \left[0, \frac{n}{n+1}\right]$,n $\in \mathbb{N}$ and $0 < q < p \leq 1$, we have

$$(i)\ V^*_{n,p,q}(e_1 - x; x) = \left[\left(\frac{n+1}{n}\right)\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right) - 1\right]x$$

$$(ii)\ V^*_{n,p,q}((e_1 - x)^2; x) = \left[x\left(\left(\frac{n+1}{n}\right)\frac{[n]_{p,q}}{[n+1]_{p,q}} - 1\right)\right]^2 + \left(\frac{n+1}{n}\right)^2 \frac{p^{n-1}x[n]_{p,q}}{[n+1]^2_{p,q}}\ x\ \left(\frac{n}{n+1} - x\right)$$

Proof. The part (i) follows directly from Lemma 2.3. Again from Lemma 2.3 and by linearity of the operators $V^*_{n,p,q}$we obtain

$$V^*_{n,p,q}((e_1 - x)^2; x)$$
$$= V^*_{n,p,q}(e_1^2; x) - 2x\, V^*_{n,p,q}(e_1; x) + x^2 V^*_{n,p,q}(e_0; x)$$
$$= \left(\frac{n+1}{n}\right)^2 \left(\frac{[n]_{p,q}[n-1]_{p,q}}{[n+1]^2_{p,q}}\right)qx^2 + \left(\frac{n+1}{n}\right) \times \left(\frac{p^{n-1}[n]_{p,q}}{[n+1]^2_{p,q}}\right)x - 2x^2\left(\frac{n+1}{n}\right)\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right) + x^2 \quad (2.3)$$

We know that $q[n-1]_{p,q} = [n]_{p,q} - p^{n-1}$, thus from (2.3)

$$V^*_{n,p,q}((e_1 - x)^2; x)$$
$$= \left(\frac{n+1}{n}\right)^2 \frac{[n]_{p,q}\left([n]_{p,q} - p^{n-1}\right)}{[n+1]^2_{p,q}}x^2 + \left(\frac{n+1}{n}\right) \times \left(\frac{p^{n-1}[n]_{p,q}}{[n+1]^2_{p,q}}\right)x - 2x^2\left(\frac{n+1}{n}\right)\left(\frac{[n]_{p,q}}{[n+1]_{p,q}}\right) + x^2$$
$$= \left(\frac{n+1}{n}\right)\left(\frac{p^{n-1}[n]_{p,q}}{[n+1]^2_{p,q}}\right)x - \left(\frac{n+1}{n}\right)^2 \frac{[n]_{p,q}p^{n-1}}{[n+1]^2_{p,q}}x^2 + \left[\left(\left(\frac{n+1}{n}\right)\frac{[n]_{p,q}}{[n+1]_{p,q}} - 1\right)x\right]^2$$
$$= \left(\frac{n+1}{n}\right)^2 \frac{p^{n-1}[n]_{p,q}x}{[n+1]^2_{p,q}}\left(\frac{n}{n+1} - x\right) + \left[x\left(\left(\frac{n+1}{n}\right)\frac{[n]_{p,q}}{[n+1]_{p,q}} - 1\right)\right]^2$$

□

3. MAIN RESULTS

In this section we give a convergence theorem , proved with the help of well known Korovkin's theorem. We also give the rate of convergence via modulus of continuity.

Let$f \in C[0,a]$. The modulus of contintuity of *f* denoted by $\omega(f,\delta)$ is defined to be

$$\omega(f,\delta) = \sup_{|s-x|<\delta} \{|f(s) - f(x)|\}, s, x \in [0,a]$$

The modulus of continuity of the function *f* in C[0,*a*] gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$. It is well known that a necessary and sufficient condition for a function f to be in C[0,*a*] is

$$\lim_{\delta \to 0} \omega(f,\delta) = 0.$$

It is also well-known that for any $\delta > 0$ we have

$$|f(s) - f(x)| \leq \omega(f,\delta)\left(\frac{|s-x|}{\delta} + 1\right) \quad (3.1)$$

Remark 3.1. For $q \in (0,1)$ and $p \in (q,1]$ we have

$$\lim_{n\to\infty} [n]_{p,q} = \frac{1}{p-q}$$

In order to ensure the convergence properties of the operators $V^*_{n,p,q}$ we take sequences $q_n \in (0,1)$ and $p_n \in (q_n,1]$ such that

$$\lim_{n\to\infty} p_n = 1 \quad and \quad \lim_{n\to\infty} q_n = 1$$

So that$[n]_{p_n,q_n} \to \infty$ as $n \to \infty$.

Theorem 3.2. Let $q = (q_n)$ and $p = (p_n)$ satisfying $0 < q_n < p_n \leq 1$ such that $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$. Then for each $f \in C\left[0, \frac{n}{n+1}\right]$

$$\lim_{n\to\infty} V^*_{n,p_n,q_n}(f; x) = f \quad (3.2)$$

uniformly on$\left[0, \frac{n}{n+1}\right]$.

Proof. By the well known Korovkin's theorem, in order to prove convergence of the sequence of operators V^*_{n,p_n,q_n} it is enough to prove the following conditions

$$\lim_{n\to\infty} V^*_{n,p_n,q_n}(e_i; x) = x^i, \mathrm{i} = 0,1,2$$

uniformly on$\left[0, \frac{n}{n+1}\right]$.

Clearly from Lemma 2.3

$$\lim_{n\to\infty} V^*_{n,p_n,q_n}(e_0; x) = 1$$

From the fact $[n+1]_{p_n,q_n} = p_n^n + q_n[n]_{p_n,q_n}$, we have

$$\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = \frac{[n]_{p_n,q_n}}{p_n^n + q_n[n]_{p_n,q_n}}$$

Since $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = \lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{\left(p_n^n + q_n[n]_{p_n,q_n}\right)} = 1 \quad (3.3)$$

Hence

$$\lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} = 0 \quad (3.4)$$

Also

$$\frac{q_n[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} = \frac{[n]_{p_n,q_n}\left([n]_{p_n,q_n} - p_n^{n-1}\right)}{[n+1]^2_{p_n,q_n}}$$
$$= \left(\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^2 - \frac{[n]_{p_n,q_n}p_n^{n-1}}{[n+1]^2_{p_n,q_n}} \quad (3.5)$$

From (3.3) , (3.4) and (3.5) we obtain

$$\lim_{n\to\infty}\frac{q_n[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}=1 \qquad (3.6)$$

Thus from Lemma 2.3, on using (3.3) , (3.4) and (3.6) we get

$$\lim_{n\to\infty} V^*_{n,p_n,q_n}(e_1;x)=x$$

$$\lim_{n\to\infty} V^*_{n,p_n,q_n}(e_2;x)=x^2 \qquad \square$$

Next we give rate of convergence of these new operators using modulus of continuity of the function.

Theorem 3.3. Let $f\in C\left[0,\frac{n}{n+1}\right]$, $n\in\mathbb{N}$, then

$$\left|V^*_{n,p,q}(f;x)-f(x)\right| \le 2\,\omega(f,\delta_n)$$

Where

$$\delta_n = x\left|\left(\frac{n+1}{n}\right)\frac{[n]_{p,q}}{[n+1]_{p,q}}-1\right| + \left(\frac{n+1}{n}\right)\frac{1}{[n+1]_{p,q}}\sqrt{p^{n-1}[n]_{p,q}x\left(\frac{n}{n+1}-x\right)}$$

Proof. From (3.1) we obtain

$$\left|V^*_{n,p,q}(f;x)-f(x)\right| \le \omega(f,\delta)\left(\frac{V^*_{n,p,q}(|s-x|;\,x)}{\delta}+1\right)$$

By Cauchy's inequality and Lemma 2.3 and 2.4 we have

$$\left|V^*_{n,p,q}(f;x)-f(x)\right| \le \omega(f,\delta)\left(\left\{\frac{V^*_{n,p,q}((s-x)^2;x)}{\delta}\right\}^{\frac{1}{2}}+1\right)\left(V^*_{n,p,q}(e_0;x)\right)^{\frac{1}{2}}$$

$$= \omega(f,\delta)\left[1+\frac{1}{\delta}\left(\left(x\left(\left(\frac{n+1}{n}\right)\frac{[n]_{p,q}}{[n+1]_{p,q}}-1\right)\right)^2 + \left(\frac{n+1}{n}\right)^2\frac{p^{n-1}[n]_{p,q}x}{[n+1]^2_{p,q}}\left(\frac{n}{n+1}-x\right)\right\}^{\frac{1}{2}}\right]$$

Using $(a^2+b^2)^{\frac{1}{2}} \le |a|+|b|$, we obtain

$$\left|V^*_{n,p,q}(f;x)-f(x)\right| \le \omega(f,\delta)\left[1+\frac{1}{\delta}\left(x\left|\left(\frac{n+1}{n}\right)\frac{[n]_{p,q}}{[n+1]_{p,q}}-1\right|+\left(\frac{n+1}{n}\right)\times\frac{1}{[n+1]_{p,q}}\sqrt{p^{n-1}[n]_{p,q}x\left(\frac{n}{n+1}-x\right)}\right)\right]$$

On choosing

$$\delta=\delta_n = x\left|\left(\frac{n+1}{n}\right)\frac{[n]_{p,q}}{[n+1]_{p,q}}-1\right|+\left(\frac{n+1}{n}\right)\frac{1}{[n+1]_{p,q}}\times\sqrt{p^{n-1}[n]_{p,q}x\left(\frac{n}{n+1}-x\right)}$$

we get the desired result.

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